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A HISTORICAL AND CRITICAL REVIEW OF MATHEMATICAL SCIENCE

EDITED BY

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BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY

THE SEVENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Seventh Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY was held at Columbia University, New York City, beginning on the afternoon of Wednesday, June 27, 1900, and extending through the two following days. As usual, the meeting was held in connection with that of the American association for the advancement of science, the Society being one of sixteen scientific bodies which have responded to the invitation of the Association to meet simultaneously with it. A considerable mutual advantage naturally arises from the arrangement. Attendance is facilitated and increased; broader interests are suggested; and a sense of scientific solidarity is experienced. The affiliated societies lose nothing of their individuality or independence, their relation to the Association not necessarily involving any more than mere simultaneous meeting. Plans for a closer official connection are, however, under consideration; it may be found advisable to give the societies, as such, a representation in the councils of the Association. Such a federation would undoubtedly be beneficial in many respects. Meanwhile, a cordial spirit of coöperation prevails between the societies and the sections of the Association. The Mathematical Society devoted the two sessions of Thursday to a joint meeting with Section A, and the other sessions of the two bodies were arranged for mutual convenience of attendance. The Society contributed to the officers of the Association their common president, Professor R. S. Woodward, and to Section A the vice-president, Professor Asaph Hall, Jr., the secretary, Dr. W. M. Strong, and the vice-president elect, Professor James McMahon.

The early date of the meeting conflicted to some extent with the academic duties of many members, and reduced the period of preparation and accumulation of ma-

terial from four to two months. Nevertheless, the occasion was in all respects one of the most successful in the history of the Society. The attendance reached the highest point previously recorded, the following fifty-six members having been registered as present:

Dr. W. G. Bullard, Dr. A. S. Chessin, Professor F. N. Cole, Dr. J. V. Collins, Professor L. L. Conant, Professor E. S. Crawley, Professor E. W. Davis, Dr. W. S. Dennett, Professor L. E. Dickson, Professor W. P. Durfee, Dr. J. C. Fields, Professor H. B. Fine, Professor T. S. Fiske, Mr. W. B. Ford, Rev. J. G. Hagen, Miss Carrie Hammerslough, Professor James Harkness, Mr. H. E. Hawkes, Dr. A. A. Himowich, Professor E. W. Hyde, Dr. S. A. Joffe, Mr. C. J. Keyser, Professor P. A. Lambert, Dr. James Maclay, Professor W. H. Maltbie, Professor Mansfield Merriman, Professor W. H. Metzler, Dr. G. A. Miller, Professor E. H. Moore, Professor F. Morley, Dr. D. A. Murray, Professor A. B. Nelson, Professor Simon Newcomb, Professor H. B. Newson, Professor W. F. Osgood, Professor Anna H. Palmié, Mr. J. C. Pfister, Professor James Pierpont, Dr. M. B. Porter, Dr. E. D. Roe, Professor F. H. Safford, Professor T. J. J. See, Professor P. F. Smith, Dr. Virgil Snyder, Dr. F. H. Stecker, Professor Ormond Stone, Dr. W. M. Strong, Professor J. H. Tanner, Professor C. L. Thornburg, Miss Mary Underhill, Professor J. H. Van Amringe, Professor L. A. Wait, Professor H. S. White, Professor C. B. Williams, Professor R. S. Woodward, Professor J. W. A. Young.

Ex-President Simon Newcomb presided at the opening of the first session, on Wednesday afternoon, afterward yielding the chair to Vice-President E. H. Moore, relieving President R. S. Woodward, who was also president of the Association. Professor H. S. White, Professor E. W. Hyde, and the Secretary were also called to the chair during the later sessions. Professor Ormond Stone presided at the joint session with Section A on Thursday morning. The Council announced the election of the following persons to membership in the Society: Mr. J. L. Coolidge, Harvard University; Professor Peter Field, Carthage College; Mr. F. A. Giffin, University of Colorado; Wm. J. Greenstreet, M.A., Stroud, England; Mr. L. L. Locke, Fredonia, Pa.; Professor J. E. Manchester, Vincennes University; Professor Wm. J. Vaughn, Vanderbilt University. Six applications for membership were reported. The Council decided to set apart the life membership fund, now amounting to $600, as a special fund to be devoted to such purpose as may hereafter be designated.

The following papers were read at this meeting:

(1) Dr. A. S. CHESSIN: "On the motion of a top, taking into account the rotation of the earth."

(2) Professor F. MORLEY: "On a mechanism for drawing trochoidal and allied curves."

(3) Mr. H. W. KUHN: "Theorems on imprimitive groups" (preliminary communication).

(4) Dr. H. E. TIMERDING: "Some remarks on tetrahedral geometry."

(5) Professor H. B. NEWSON: "On singular transformations."

(6) Dr. VIRGIL SNYDER: "On a special form of annular surfaces."

(7) Professor F. MORLEY: "On the rational quartic curve in space."

(8) Professor PAUL GORDAN: "Die Hesse'sche und die Cayley'sche Curve."

(9) Mr. H. E. HAWKES: "On hyper-complex number systems."

(10) Professor MAXIME BÔCHER: "Application of a method of d'Alembert to the proof of Sturm's theorem of comparison."

(11) Miss I. M. SCHOTTENFELS: "On groups of order 8!/2."

(12) Professor P. F. SMITH "On surfaces sibi-reciprocal under those contact transformations which transform spheres into spheres."

(13) Professor E. H. MOORE: "A simple proof of the fundamental Cauchy-Goursat theorem."

(14) Professor W. F. OSGOOD: "On the existence of the Green's function for the most general simply connected plane region."

(15) Dr. J. V. COLLINS: "Quaternions and spherical trigonometry."

(16) Professor J. MCMAHON: "Kelvin's treatment of instantaneous and permanent sources extended to certain cases in which a source is in motion."

(17) Dr. F. R. MOULTON: "Oscillating satellites."

(18) Miss B. E. GROW: "The reduction of binary quantics to canonical forms by linear transformation."

(19) Dr. M. B. PORTER: "Note on geometry on the non-singular cubic."

Mr. Kuhn's paper was presented to the Society through Dr. G. A. Miller, Dr. Timerding's and Professor Gordan's through Professor H. S. White, Miss Schottenfels's through the Secretary, and Miss Grow's through Professor H. B.

Newson. In the absence of the authors, Mr. Kuhn's paper was read by Dr. Miller, Dr. Timerding's and Professor Gordan's by Professor White, Professor McMahon's by Dr. Snyder, Dr. Moulton's by Professor Moore, and Miss Grow's by Professor Newson. The papers of Professor Bôcher, Miss Schottenfels, and Dr. Porter were read by title. The papers of Dr. Timerding and Miss Schottenfels appeared in the BULLETIN for July, 1900. Abstracts of the other papers are given below.

The Friday afternoon session was mainly devoted to a discussion of the following question:

What courses in mathematics should be offered to the student who desires to devote one-half, one-third, or one-fourth of his undergraduate time to preparation for graduate work in mathematics?

The discussion was opened by the following papers:

Professor E. H. MOORE: "Certain fundamental ideas which should be emphasized throughout the undergraduate course."

Professor J. HARKNESS: "The importance of some preliminary training in applied mathematics"; "Courses in differential calculus and differential equations."

Professor W. F. OSGOOD: "Shall we try to place the calculus as early as possible?" "How early in the course may the lecture method be used with profit?" "Are the best results in graduate work secured from students who have devoted most of their undergraduate time to mathematics or from those who have combined a fair amount of mathematics with a more general culture?"

Professor F. MORLEY: "The theory of equations; spherical trigonometry; geometric conics."

Professor J. W. A. YOUNG: "Collegiate preparation for the teaching of mathematics in secondary schools."

A separate report of the discussion, prepared by Professor W. H. Maltbie, is printed in the present number of the BULLETIN, pages 14–24.

Professor Chessin's paper is an application of the general theory of relative motion as presented by the author in his paper "On relative motion" in the *Transactions*, volume 1, number 2, pages 116–169. The top here considered is a body of revolution with one point of its axis fixed on the surface of the earth. The problem of this so called Foucault top has not heretofore been solved and, indeed, it was thought that it did not lend itself to a solution. (See the memoir "On the application of Lagrange's method to

problems of relative motion," by Ph. Gilbert.) The motion of the earth is considered as a perturbation of that motion which the top would have if the earth did not rotate. Among the effects produced by the rotation of the earth may here be mentioned more especially that which appears in the precessional motion of the top. The precession may be expressed by the formula

$$\frac{a}{N} - C\omega,$$

where N and ω denote, respectively, the number of revolutions of the top per time unit and the angular velocity of rotation of the earth; a and C are certain coefficients which depend on the initial deviation of the top from a vertical position, on the moments of inertia, etc. The sense of the revolutions of the top being fixed, say from left to right for an observer along the axis of the top with his feet at the fixed point of the axis, the precessional motion of the top may take place from left to right, or on the contrary from right to left, or indeed there may be no precession at all, according as Na is greater than, less than, or equal to $a:C$. These results are in harmony with similar conclusions reached by the author in the case of Foucault's pendulum, previously discussed.

The model shown by Professor Morley was an embodiment of the following idea: A rod OA is pivoted at O and a rod AP is hinged at A to the former rod. When such a system moves in a plane, the rods having constant or proportional angular velocities, then P describes the general trochoid. It is seen at once from the triangle of velocities that the normal at P meets OA at a point N fixed in OA. Conversely, then, if a rod PN passes through a point of OA, and has a toothed wheel at P, whose axis is PN, the wheel will describe a trochoid on a drawing-board. By adjustments the lengths OA and NA can be altered, so that the one apparatus traces all trochoids within a certain range; though not all possible trochoids. Further, by adjusting the angle which the axis of the wheel makes with PN, we can draw the curve which cuts at a given angle all the ring of trochoids obtained by rotating a trochoid about its center.

The groups which can be represented as imprimitive groups only when the degree equals the order are divided by Mr. Kuhn into those that can be represented transitively only in the regular form and those that can also

be represented as primitive groups which are not regular. The latter are shown to be the non-cyclical groups of order pq (p and q being different prime numbers). The former include (a) the abelian groups; (b) the Hamiltonian groups; and (c) those groups that contain non-invariant subgroups, each of which includes an invariant subgroup. The last are omitted by Dyck, *Mathematische Annalen*, volume 22. The paper includes several theorems relating to the construction of imprimitive groups whose degree is the product of two prime numbers and which are isomorphic to any transitive subgroup of the metacyclic group whose degree is one of the primes. By means of these theorems most of the imprimitive groups of degree fifteen, for example, can be written down at once.

Singular transformations in certain projective groups of the plane have been determined by Study, Taber, Rettger, and others. The object of Professor Newson's paper is to determine the character and type of these singular transformations. He finds in the case of the special linear homogeneous group in the plane, Study's case, that the singular transformations are of type II., while the continuous subgroups are of types I. and V. The singular transformations are characterized by a variable a and a constant value of k, viz, $k = -1$. These values of the parameters do not satisfy the necessary and sufficient relation for a one parameter group of type II., viz, $k = a^e$. (See BULLETIN, volume 6, number 10, page 438.) These singular transformations are obtained by combining the single involution perspective transformation of the group with all those of type V. in the group.

In Dr. Snyder's paper the equations of annular surfaces having two spherical directrices are first derived by means of Lie's equations connecting line sphere space for both the cases in which the directrices are distinct and coincident. A particular case is next given showing that a non-reducible cone of order 6 exists, all of whose lines of curvature are of order 6. On transforming this surface into line space, making a linear transformation upon its points, and then putting the new scroll back into an annular surface, the latter will be of order 12, not of revolution, but will have plane lines of curvature of order 6. More generally, by revolving any plane curve about any line in the plane, transforming the surface generated into a scroll, making a certain projection of the scroll, then putting the new scroll into sphere space

an annular surface is obtained having plane lines of curvature whose order is half that of the surface—the latter not being of revolution. The scrolls are also of particular form; their asymptotic lines are of order only half that of the general scroll of the same type.

Professor Morley's second paper is in abstract as follows: It is known that the bitangent planes of a rational space quartic R_4 touch a quadric, which contains the tangents of the curve at the flexes or points of contact of the four stationary planes. Any tangent plane of this quadric cuts R_4 at four points U, such that the Hessian points H of U also lie in a plane. This plane again touches the quadric. Any tetrad $U + \lambda H$ of R_4 lies in a plane which touches the quadric. All the planes form a pencil whose axis is a generator of the quadric, of the system to which the flex lines belong. Among the planes of the pencil there are two which cut R_4 in a self apolar tetrad, three which cut R_4 in apolar pairs of points and three bitangent planes. The one set of three planes is the cubicovariant of the other; the two planes form the common Hessian of the two sets of three planes.

Professor Gordan's paper, which will be published in the *Transactions*, is in brief an exhaustive discussion of the curves of order or class 3, which have for their Hessian or Cayleyan a given curve of the same order or class. The author shows that these number 3 in general, but in special cases 1 or ∞; for each case the types are determined explicitly.

The point of departure for Dr. Hawkes's paper was furnished by a consideration of certain criticisms of Benjamin Peirce's Memoir on linear associative algebra, by Study, Molien, Cayley, and others. The paper shows how Peirce's methods can be naturally extended so as to arrive very directly at an enumeration of all number systems in less than six units, which are inequivalent, non-reciprocal, irreducible, and possess moduli.

Of the many theorems contained in Sturm's famous memoir of 1836, the two Theorems of Comparison may be regarded as most fundamental. Professor Bôcher has shown on other occasions (cf. BULLETIN, April 1898 and Dec. 1899, p. 100) how the methods which Sturm used for establishing these theorems can be thrown into rigorous form. In the present paper a simpler and more direct method of establishing these

theorems is given along lines suggested by passages, to which Professor H. Burkhardt kindly called the author's attention, in one of d'Alembert's papers on the vibration of strings (*Memoirs of the Berlin Academy*, volume 70 (1763), p. 242). The fundamental idea involved is the reduction of the linear differential equations of the second order to Riccati's equations by the well known change of dependent variable.

The problem discussed by Professor Smith is a generalization of the question treated by Moutard in 1864, viz., to find properties of surfaces enveloped by spheres intersecting a fixed sphere at right angles. The generalization is this: The surfaces to be studied are enveloped by spheres intersecting a fixed sphere at any constant angle. As distinguished from the problem discussed by Moutard, the question is one of the higher spherical geometry, in that the algebraic sign of the radius of a sphere must be regarded. This amounts to distinguishing between the two sides of a surface, as given, *e. g.*, by the two directions of the normal; and this distinction is indicated by the term, "oriented surface." The treatment of the problem is purely geometrical, being based upon a certain simple contact transformation, here defined for the first time. Adopting Sophus Lie's terminology, "linear spherical complex," for the aggregate of ∞^3 oriented spheres intersecting a fixed fundamental oriented sphere under constant angle, this definition is as follows: Given an oriented sphere S, then all ∞^2 oriented spheres tangent to S and belonging to a spherical complex A are tangent to a second oriented sphere S', and S is said to transform into S' by inversion in the complex A. The transformation is involutory. This general inversion depends upon five essential parameters, and includes as particular cases inversion in a sphere, the parallel transformation, symmetry in a plane, and the transformation by reciprocal directions originally defined by Laguerre in 1881. The ∞^3 points of space constitute a particular spherical complex Ω, and inversion in Ω merely turns over an oriented plane.

The relation of this general inversion to the general sphere-sphere contact transformation of fifteen essential parameters is determined by the theorem: The latter transformation is compounded either of a general inversion, a spherical inversion and a displacement, or these three compounded with inversion in Ω. Various theorems on the general inversion are established, *e. g.* (1) A general inversion A is uniquely compounded of a parallel transformation

D, a spherical inversion I and the same parallel transformation D, *i. e.*, $A = DID$. (2) A general inversion A is compounded in two ways of a general inversion A_0, the inversion in Ω, and the same inversion A_0, or $A = A_0 \Omega A_0$. The last theorem leads to a representation of the ∞^3 oriented spheres of the complex A by the ∞^3 points of space, in that the former transform into the latter by inversion in A_0. This representation is fundamental for the point of view adopted in the paper. The following result is now established:

A surface F enveloped by ∞^2 (∞^1) spheres belonging to the spherical complex A is transformed by A_0 into a non-oriented primitive surface (curve) $f(c)$, regarded as the locus of its points. The order and class of F are equal, and its entire section with the plane at infinity is along the imaginary circle. The centers of these spheres of A lie on a surface F_0 (curve C_0), which is the transformed of $f(c)$ by a birational transformation whose fundamental curve is the imaginary circle. The characteristics of F, F_0, and C_0 are easily determined from those of f and c. The intersection of the fundamental sphere of A and the primitive is a double line (point) on F, while the remaining intersection of F and the fundamental sphere is a line of curvature on F.

Particular interest attaches to the case where the primitive is a general cyclide without nodes. Then F is the so-called surface of singularities of the general quadratic spherical complex, and the following results are established:

The surface of singularities of the general quadratic spherical complex is of order and class 24, and contains the imaginary circle at infinity twelve times. It is sibi-reciprocal under six commutative general inversions A_i, with each of which is associated a cyclide f_i, from which as primitive the surface may be derived. The surface contains six double lines, sphero-quartics, the intersections of the f_i and the fundamental spheres, and is the envelope in six ways of ∞^2 spheres belonging to an A_i whose centers lie upon a quartic surface F_i containing a double conic. The six F_i are confocal, being inscribed in the minimum developable of order 24 circumscribing F, which is also the singular minimum developable of that surface. Thus the focal lines of the F_i are the focal lines of F. The surface F is enveloped by six systems of bitangent planes each belonging to a developable of order 8 and circumscribed to a sphere concentric with the corresponding fundamental sphere along a sphero-quartic. The lines of curvature of F are algebraic, being the transformed of the lines of curvature of the prim-

itive cyclides f_i by the six inversions which transform the f_i into F.

When the primitive is a circle (or straight line), the surface F is the cyclide of Dupin. This leads to a very simple derivation of the properties of this surface. Some special cases departing from the general results are discussed. The problem is taken up also for the plane, the case in which the primitive is a bicircular quartic giving results analogous to that just given for the cyclide, the order and class being 16 and each circular point octuple.

It is pointed out that in papers on subjects related to that under discussion the method adopted has been that of deriving results from known theorems in line geometry by the line-sphere transformation of Sophus Lie. To this there are two objections: (1) Spherical geometry is essentially metric; and those theorems in line geometry which lead by this correspondence to such results as are developed above are not *a priori* evident. (2) The correspondence fails except for the case of three dimensions. The discussion of this paper, operating entirely in the realm of spherical geometry, gives results which may be immediately generalized for n dimensions. Thus, *e. g.*, the relation of the problem treated in the paper to the minimum projection of points in a space of $n+1$ dimensions into the oriented spheres of space of n dimensions is determined.

Goursat's memoirs [*Acta Mathematica*, volume 4 (1884); *Transactions*, volume 1 (1900)] in proof of Cauchy's integral theorem

$$\int_C f(z)\,dz = 0$$

without assumption as to the continuity of the derivative $f'(z)$ on the closed region bounded by the curve C of integration lay deeper foundations for the Cauchy-Riemann theory of functions of a complex variable. One finds an abstract of these memoirs in the BULLETIN for June, 1899, pp. 427–429. Goursat sets out by a direct process to evaluate the integral in question. In Professor Moore's paper, which will appear in the *Transactions*, the integral is by an indirect process proved to have the value zero. The essential elements of the proof are those of Goursat's first paper; by the modification indicated one avoids the necessity of using the lemma to the proof of which Goursat's second paper is devoted.

The following is a summary of Professor Osgood's paper,

which appeared in full in the July number of the *Transactions:* The problem of mapping the interior of a simply connected plane region T conformably on the interior of a circle depends for its solution on the proof of the existence of the Green's function corresponding to T. Hitherto the existence of this function has been established for regions T bounded by a finite number of pieces of analytic curves, and, more generally, for regions bounded by a finite number of pieces of regular curves, *i. e.*, curves whose tangent turns continuously as a moving point p traces out the curve. That the boundary of a general simply connected region T may be of an essentially more complicated character is shown by the following example: On the axis of abscissas assume at pleasure a set of points perfect but not dense throughout any interval whatsoever. Erect at each point of this set a perpendicular of unit length, drawn in the direction of the positive axis of ordinates. The region T consists then of the points of the upper half plane that remain after these perpendiculars have been cut out. Thus the boundary of T will in this case have a positive content if the content of the assumed set of points on the axis of abscissas was positive. Moreover, there now exist points of the boundary that cannot be approached along a continuous curve, namely, any interior point of a perpendicular about which perpendiculars cluster from both sides.

In the present paper the existence of the Green's function for the most general simply connected region T is established by methods due to Poincaré, and thus it appears that the interior of any such region whatsoever can be transformed conformably on the interior of a circle.

Dr. Collins deduced the fundamental formulæ of spherical trigonometry by means of quaternions, employing the sine-cosine, in place of the scalar-vector notation.

The conception of a moving source treated in Professor McMahon's paper was suggested by the following problem set by a physicist who wished to use the solution in connection with an experiment: A source of heat (such as a candle) starts at a given point of a very long straight wire, and is moved with uniform velocity along the wire; it is required to find the temperature at any assigned point of the wire at any future time; and especially to compare the temperature gradient in front of the candle with that behind the candle, when these gradients have become practically stationary relatively to the moving source. The methods

of Fourier are not directly applicable, but a solution can be obtained by using Kelvin's expression for the effect of an instantaneous source of heat placed at a given point of an infinite rod, and superposing the temperature effects due to a succession of such instantaneous sources placed at successive points of the line. The solutions for a finite rod with various terminal conditions are then given; and the corresponding problems for an infinite plane and for a finite rectangle are solved. The results can be extended to three dimensions.

Lagrange found particular solutions of the problem of three bodies of two types: (*a*) the three bodies lie always in a straight line; (*b*) the three bodies are always at the vertices of an equilateral triangle. These solutions still exist when one of the bodies becomes *infinitesimal*, and when the two finite bodies move in *circles*. The problem is to find whether solutions exist in which the infinitesimal body describes closed curves in the vicinity of the points defined by the Lagrangian solutions. In Dr. Moulton's discussion use is made of the methods employed by Poincaré in his prize memoir in the *Acta Mathematica*, volume 13. The results obtained have points of contact with Darwin's memoir in the *Acta Mathematica*, volume 21. The principal results are the following: Let $1-\mu$ and μ represent the masses of the finite bodies. Let $\mu = \mu_0 + \lambda$. There is a periodic solution for every μ_0 in which the infinitesimal body moves in the plane of the motion of the finite bodies, in the vicinity of each one of the Lagrangian straight line solutions, and in which the coordinates are expansible as power series in λ, vanishing with λ, and with periodic coefficients, the periods depending upon μ_0. Thus there is an infinite number of orbits vanishing with $\mu = \mu_0$. In the case of the equilateral triangular points in the plane there are two infinities of orbits with different periods vanishing with $\mu = \mu_0$, *if* $\mu_0 < .3086$. In the vicinity of the equilateral triangular points in space of three dimensions there is an infinity of solutions, with periods differing from the other two, which vanish with $\mu = \mu_0$ for *every value of* μ_0.

Miss Grow's paper is in abstract as follows: The transformation

$$T: \quad \begin{aligned} z_1 &= \lambda_1 z_1' + \mu_1 z_2', \\ z_2 &= \lambda_2' z_1' + \mu_2 z_2', \end{aligned}$$

converts $f=e_x^n$ where the coefficients of f are independent, into the form

$$f'(z') = e_\lambda^n z_1'^n + \binom{n}{1} e_\lambda^{n-1} e_\mu z_1'^{n-1} z_2' + \cdots + e_\mu^n z_2'^n.$$

If it is possible to transform the quantic by T so that any two assigned coefficients, for example the $(n-r+1)$th and $(k+1)$th, shall vanish simultaneously, the necessary and sufficient condition is

$$e_\lambda^r e_\mu^{n-r} = 0, \quad e_\lambda^{n-k} e_\mu^k = 0$$
$$(r = 1, 2, \cdots, \quad n-1; \quad k = n-1, \quad n-2, \cdots, 1; \quad k \neq n-r).$$

Consider (1) $e_x^r e_\mu^{n-r}$, (2) $e_x^{n-k} e_\mu^k$ and (3) $e_x^{n-r} e_\lambda^r$, (4) $e_x^k e_\lambda^{n-k}$.

If (1) and (2) vanish simultaneously, their resultant $R_{n-r,k}$ vanishes; also if (3) and (4) vanish simultaneously, their resultant $R_{r,n-k}$ vanishes. $R_{n-r,k}$ and $R_{r,n-k}$ each have $[(n-r)(n-k)+rk]$ zeros in the plane; therefore (1) and (2) will vanish simultaneously when μ is a root of $R_{n-r,k}$, and (3) and (4) when λ is a root of $R_{r,n-k}$. f is always a factor of $R_{n-r,k}$ and $R_{r,n-k}$, or $R_{n-r,k} = fC_{n-r,k}$, and $R_{r,n-k} = fC_{r,n-k}$.

The z-points of (1) and (2) are the λ-points of $R_{r,n-k}$. Similarly the z-points of (3) and (4) are the μ-points of $R_{n-r,k}$, *i. e.*, the μ-points of $R_{n-r,k}$ and λ-points of $R_{r,n-k}$ are so related that

$$e_\lambda^{n-r} e_\mu^r = 0, \qquad e_\lambda^k e_\mu^{n-k} = 0.$$

When $r=k$, $C_{n-r,k}$ and $C_{r,n-k}$ coincide in $\overline{C}_r$. The roots of $\overline{C}_r$ go in pairs which have reciprocal relations. In the transformation T, let λ and μ be roots of $C_{r,n-k}$ and $C_{n-r,k}$ respectively and call this set of transformations $T_{n-r,k}$. Interchange λ and μ and call this set $T_{r,n-k}$. The non-singular binary quantic can be linearly transformed by the $[(n-r)(n-k)+kr-n]$ transformations $T_{n-r,k}$ so that its $(n-r+1)$th and $(k+1)$th terms shall vanish, and by $T_{r,n-k}$ so that its $(n-k+1)$th and $(r+1)$th terms shall vanish. Also if λ and μ are zeros of $\overline{C}_r$ in corresponding pairs, we get two sets of transformations $\overline{T}_r$ and $\overline{T}_r'$ by means of which its $(r+1)$th and $(n-r+1)$th terms can be made to vanish in $2r(r-n)+n(n-1)$ different ways.

The theory is applied to the cubic, quartic, quintic, and sextic.

The following is a summary of Dr. Porter's paper: Consider $3m - s - 1$ arbitrary fixed points P on a non-singular cubic C_3, and $u_i = \int_{ab}^{x_i y_i} du$ the integral of first kind on C_3, (ab) being a point of inflexion. If an m-ic have a $s-1$ order contact at u_1, it will cut C_3 again at u_2, $su_1 + u_2 \equiv C$ (mod. ω, ω') where $C = \Sigma u_i$ at the points P. The Schliessungsproblem thus suggested yields at once a proof of Fermat's theorem $a^n - a \equiv 0$ (mod. n (prime)) and the generalized form of the theorem $F(a, n) \equiv 0$ (mod. n). When $m = 1$, $s = 2$, we have systems of closed polygons. In case the polygon is a triangle, the equation of C_3 referred to it may be written

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + 2\eta = 0.$$

The twenty-four in-circumscribed triangles thus determined fall into four groups, each associated with an inflexion triangle, and each triangle of a group six ways perspective with its associate inflexion triangle. This configuration of inflexion triangles and in-circumscribed triangles presents numerous interesting geometrical properties.

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THE UNDERGRADUATE MATHEMATICAL CURRICULUM.

REPORT OF THE DISCUSSION AT THE SEVENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE final session of the Seventh Summer Meeting of the Society was devoted to an organized discussion of the following question:

What courses in mathematics shall be offered to the student who desires to devote one-half, one-third, or one-fourth of his undergraduate time to preparation for graduate work in mathematics?

The following topics were also suggested as a general basis of discussion:

How early in the course may the lecture method be used with profit?

How can the undergraduate best be trained to use the library?

How shall the history of the subject be presented?

Shall we try to place the calculus as early as possible?

Where shall we place the solid analytics? modern geometry? projective geometry?

Shall spherical trigonometry and geometric conics be given a place in the curriculum?

How much of the theory of equations may be presented with profit in the freshman year?

What courses in applied mathematics are needed as preparation for graduate work in pure mathematics?

Shall work in differential equations be merely a problem course, or shall it take up the theoretical side of the question? If the latter, to what extent?

At what points shall the work intended to fit the student for graduate work differ from that intended to fit him for secondary school teaching?

Are the best results in graduate work secured from students who have devoted most of their undergraduate time to mathematics, or from those who have combined a fair amount of mathematics with a more general culture?

Shall the undergraduate school attempt to attract the student to graduate work by offering elementary courses in the more advanced topics, or shall it confine itself to fundamental work in algebra, analytics, and calculus?

The principal papers are given in abstract below.

Professor Moore.

If the student is to be properly prepared for graduate work the teacher must be himself familiar with modern mathematics; and the fundamental modern ideas and methods must give form to his work as an instructor of undergraduates. Thus the teacher should have clearly in mind the notions of the pure analysis—of the sequence of positive integers as a complete set, of the synthetic determination of all real numbers from this sequence of integers and of the complex variable as a double real variable. Similarly he should have the (usual) idea of (a) geometry as built upon a body of basal notions and axioms.

In general, mathematics should be regarded as divided into a number of distinct but closely related sciences. Each of these sciences consists of a body of sharply defined and inter-related ideas as foundation, with a superstructure consisting of results deductively secured from these funda-

mental ideas. For every science the question of or the condition for its existence as a body of non-conflicting statements is of prime importance, and especially by these existence considerations are the various sciences related and organized into higher sciences.

Intuition is by no means to be excluded from mathematical work. On the contrary, in every way let intuition be kept alive and active, but when a new idea is introduced as an intuition let it be sharply defined with the understanding that in so far as subsequent work depends on that intuition a new branch of mathematics is being originated.

In illustration: It is questionable whether the undergraduate course of a student who is not specializing extremely in mathematics may advisably contain the existence theory of the real irrational number in its ultimate form as a part of pure analysis. The student, however, may be led to grant the fundamental properties of the system of all real numbers as intuitively true for the system of all points of a straight line. On the basis of these assumptions the student develops a geometric analysis which will only later in pure analysis receive a deeper foundation, the character of which however should at once be briefly indicated. Similarly, in dealing with the complex variable in the Gauss plane, the radius and the angle may be regarded as simple ideas, although they are to appear as much more complicated creations in pure analysis.

Professor Harkness.

In these days of extreme specialization half the college time is too little to serve as an adequate basis for later work, unless the student enters college well prepared in elementary mathematics. With improved educational machinery in the secondary schools, there is no reason why a fair number of our college freshmen should not have had some preliminary training in analytic geometry, differential calculus, statics, and dynamics. This is the case in England and would be the case here, if the brighter boys were not held back in order that they may not get too far in advance of the less intelligent members of their classes.

As regards college students I urge the importance of increased attention to applied mathematics. In mathematics as in everything else there are fashions. At one time the subject of invariants, at another that of the theory of functions, and later still Lie's continuous groups have engrossed the attention of pure mathematicians; today algebraic

numbers, or briefly those branches of mathematics which we associate with the names of Kronecker, Hilbert, Hensel, and Weber, are in a central position. All this time applied mathematics have been out of fashion, but there are many signs that a change is coming and the intelligent college instructor will do well to anticipate this change as far as possible.

However this may be, it is of great importance for the pure mathematician of today that he should be well grounded in the fundamental principles of statics, dynamics, and the theory of the potential. Riemann's programme cannot be appreciated at its proper value without a firm grasp of the fundamental principles of the theory of the potential so far as it relates to Dirichlet's Problem. It would be easy to multiply examples of this kind; for instance Schottky's memoir in the 83d volume of Crelle contains Weierstrass's Lückensatz, and is therefore of the greatest interest to the mathematician concerned with algebraic functions and Abelian integrals, but papers of this kind are apt to be passed by as unintelligible unless the reader has had some preliminary training in the discussion of conjugate functions, conform representation, and the behavior of the potential in multiply connected regions.

A good early training in statics and dynamics (including dynamics of a particle and the elements of rigid dynamics) would go far to remove many of the difficulties experienced by graduate students in the reading of modern memoirs, and would widen their mental horizon in a variety of ways. In the college instruction itself such a training would throw much needed light on many of the problems of the differential and integral calculus; partial differential equations would be connected with physical problems, and the undergraduate would feel, as he hardly feels at present, that the formulæ he meets with are capable of concrete representations.

I urge the necessity of a thorough revision of college courses in differential calculus. The subject is too often taught along traditional lines; the lecturer should ask himself whether certain parts of his subject possess the same importance today that they had in Euler's time. Much might be rejected as of secondary importance in comparison with the newer developments. Our text books abound, too, with proofs that are no proofs and with extremely misleading statements. It is very desirable that such proofs should be rejected, or given only with clear indications of their limitations. A graduate student in the theory of functions

often finds it impossible to rid his mind of erroneous notions derived from the days when he first began the study of the differential calculus.

While due emphasis should be laid on the fundamental concepts of function, limit, continuity, etc., the work of the undergraduate should be lightened as far as possible. He should not be asked to differentiate x^{x^x} or to work out examples which illustrate no general theory and are unlikely to be of use to him later on. Moreover care should be exercised when he comes to applications; the simple cases are usually discussed correctly, but the writers of our text books are too apt to go further and attempt a partial and inaccurate discussion of more difficult cases. Here again it would be easy to multiply examples; it will be sufficient to refer to the criteria for maximum and minimum values and to the accounts usually given of the higher singular points on an algebraic curve.

In conclusion I desire to emphasize the great importance of thoroughness; the greatest service that can be rendered to a prospective graduate student is to give him a complete mastery over those parts of the technique of mathematics which he is certain to need at a later stage of his work.

PROFESSOR OSGOOD.

In school instruction in geometry the idea is emphasized from the beginning that certain definitions and axioms are laid down, on which the whole subject of geometry is built up by means of logical reasoning; and the pupil's reasoning power and appreciation of geometric truth are developed by means of original exercises that are given him to work out by himself. That instruction in algebra is at present far less highly developed as regards effectiveness for the purposes both of general training and of giving the pupil a correct conception of the subject of algebra itself, is due doubtless in part to the formal side of the subject. Dexterity in the manipulation of formulas is indeed important for later work; but the foundation of algebra is arithmetic, and the arithmetic source of algebraic principles ought not to be lost sight of to any such extent as is at present the case. How far improvement here in school instruction is possible I will not undertake to say, but of this I am sure, that this difficulty must be met and can be met from the start in instruction in calculus. The foundation of the calculus as we have the science to-day is arithmetic. Now it is

neither feasible nor desirable to begin a college course in calculus by first developing the number concept. I find it useful to assume the conception of a scale; more precisely, to make the assumption that, two points having been chosen at pleasure on a straight line to represent the numbers 0 and 1, every other (real) number can then be represented by a point of the line; and conversely, to every point of the line there will correspond a number. Similarly the conception of a function of a single variable or of two variables, is coextensive with the intuitive picture of a plane curve or of a surface. The fundamental principles of the calculus must be taught in a manner wholly different from that set forth in the text books if they are to become flesh and blood to the student. It is an easy method for the teacher to expound the notion of the limit in the opening lectures of the course and from there on to appeal to this conception, this mode of thinking, as if it had become a part of the mental machinery of the beginner. No greater mistake can be made. The student will indeed learn to perform differentiations and integrations; but he will not make progress in mastering the central ideas of the calculus. The notion must be set forth repeatedly throughout the whole course. A valuable aid in making this notion clear is furnished by the application of the calculus not merely to differential geometry, but (the integral as the limit of a sum having been introduced early in the first course in calculus) to problems in volumes, centers of gravity, moments of inertia, fluid pressures, gravitational attractions, etc.* It is the deduction of the formula that brings out anew each time the fundamental principle of the limit, while the problems themselves are typical applications of the calculus to physics, and interest and train alike the student who is preparing for work in pure mathematics and him who is to concern himself primarily with its applications.

I have thought it necessary to state with some fulness some of the leading features of a course in calculus, for there are still colleges in this country in which calculus means a systematic study of Williamson's text book—formal differentiations, formal integrations, and a total lack of the ideas that are central in the calculus of today. A course in calculus along the lines that I have indicated is effective in introducing the student to modern mathematics, and its notions and methods are in themselves interesting

*This part of an introductory course in calculus is well set forth by the problems in Chap. V. of Professor Byerly's Problems in Differential Calculus.

and valuable for the purpose of general training. It is best placed in the sophomore year for students who have entered college on the ordinary requirements.

As to how early the lecture method may be used with profit I may say that at Harvard the lecture method has been employed with success for over twenty years in practically all of the courses in mathematics, except in the freshman course in solid geometry. Written exercises, chiefly problems, are handed in by the student at each meeting of the course, and the instructor frequently spends some minutes on points that may have given general difficulty or require special comment.

It has been the practice at Harvard, ever since the graduate school came into existence, not to separate the graduate and the undergraduate departments as regards methods of instruction or the unrestrained admission of properly qualified students registered in the one department to the courses of the other department. Since there is free election from the freshman year on and the choice of courses is large after a student has passed beyond the elementary courses in calculus, geometry, and mechanics, it is the rule rather than the exception that the bright student, following his inclination to a special line of work in mathematics, will proceed rapidly to higher courses in this particular line; and thus a course ordinarily taken by first year graduates, say, will have seniors or even juniors in it, while on the other hand students that have paid more attention to other lines of work may not take it till a later graduate year. The advantage that is gained from this arrangement is obvious. The older students bring with them greater maturity for the work of the course and contribute to setting a higher standard, and the younger ones derive from the work they see their fellow students doing profit which they could not get from the lecture alone. In this respect the German university is especially strong. There, a student has free scientific intercourse with more advanced fellow students from the time he enters the university till he becomes himself a professor. At Göttingen, for example, there is a mathematical society composed of the university teachers, students whose attainments are measured approximately by the doctor's degree, and a goodly number of advanced students whose attainments are intermediate between those of these two classes. At its weekly meetings the current literature is discussed and lectures on current topics are given by its members. The formal meeting is followed by an informal one, usually

held at an inn or in a summer garden, at which there is abundant opportunity for free discussion of scientific topics. The advantages that accrue to the undergraduate courses in one of our colleges from the presence of a graduate school are not unlike the benefits that the younger members of this society derive from their intercourse with its older members.

PROFESSOR MORLEY.

As to the socalled geometrical conics, a little of this might wel' be tacked on to elementary plane geometry. The beginner is apt to suppose that Euclid's methods in plane geometry are good only for the line and circle. Spherical trigonometry should form not a separate course but a chapter in solid analytics. The latter is somewhat neglected in college work, owing to the difficulty which poor students find in it. The plan of joining both two and three dimensions in one book is a good one, but the books in use are too old fashioned for the class of men we are considering. As to the theory of equations in the old sense (that of Burnside and Panton), it can be and I think should be attached to elementary analytic geometry. For instance when the curve

$$y = a + bx + cx^2$$

is mastered and its slope can be found, then the curve

$$y = a + bx + cx^2 + dx^3$$

should be fully discussed. The points where the curve meets $y = 0$ would lead to the consideration of Horner's method. The general parabolic curve would then be mentioned. Taylor's theorem would make its first appearance here, as a convenient formula for changing the origin and discussing the curve near a point other than $x = 0$. Similarly the notion of covariants should grow out of geometry.

In general the notions of arithmetic and geometry with which a student enters college should be developed by interaction, say through plane and solid analytics and a course in calculus, before either the theory of arithmetic or that of geometry is considered philosophically. This is in effect a plea for the existing order of subjects, if they are presented in a way which leads up to the subjects as they stand, and satisfies both the refined views of the teacher and the common sense of the student. Thus, in the case of the calculus, the books used to begin with some tremendous generalities.

We have either to justify these, or to throw them aside, or to do both. We should do both—first have a calculus proper, of the kind dear to physicists, accurate but not abstract, carefully weeded of difficult generalities; and at a later stage a philosophical theory. To begin with the latter would to my mind be a mistake; the appeal would be to too limited a class.

Thus in my view the first half of a college course would be the same for the future mathematician as for the future physicist. And in fact there would be direct gain for the mathematician if the work were better adapted for the physicist than at present. Thus we ought not only to connect arithmetic with geometry, but also with dynamics, and perhaps with other physical subjects such as electricity, taking care that the student has the definite physical concepts to which we are appealing.

Lastly we must take hints from the technical schools, for instance with regard to drawing diagrams and the use of integrators or of abaci.

PROFESSOR YOUNG.

Considering the fundamental topic of this discussion to be:—*The work in mathematics in college: its purpose, scope, methods and relation to graduate work in mathematics*, my remarks are to be made from a special point of view, viz., when the purpose is to prepare teachers of mathematics for the secondary schools.

I. *The need of such preparation.*—It is a truism to say to this audience that secondary school teachers of mathematics should have had a collegiate education with special attention to mathematics either by electives in the college course or by some graduate study. It is equally well known to this audience that this standard is by no means generally reached. The time seems to be ripe, however, for setting up the general culture of a full college course and the special preparation of a certain minimum of mathematical attainment as norm for the educational qualifications of new appointees to the work of teaching mathematics in our secondary schools.*

II. *The preparation, when to be attained.*—However desirable some graduate work may be, the fact must be recognized that at present thenorm mentioned above represents a decided advance, and that for some time to come the majority of the secondary school teachers will have no preparation

* Cf. Resolution III of the Committee on the College Entrance Requirements, p. 30 of report submitted July, 1899.

beyond a college course. The college program should consequently include all the work needed for the minimum of preparation.

III. *The subject matter of the minimum course.*—As the minimum of mathematical attainment the following courses are suggested:—Thorough courses in plane trigonometry and college algebra, and good first courses in analytic geometry of the conic sections, differential and integral calculus, theory of algebraic equations, determinants, modern synthetic geometry, and, if possible, analytic mechanics.

IV. *Pedagogic preparation.*—While experience has shown that special training in the art of teaching mathematics is of high value, it is desirable that the recipient of this training be as mature as is feasible. The best place for the pedagogic training and instruction is therefore *after* the college course; either in the university or by some form of apprenticeship in the secondary schools. (Under favorable circumstances, helpful work of this sort could doubtless also be done in the senior year of the college.)

V. *Mathematical Independence.*—By mathematical independence I mean a degree of mastery of subject matter and methods such that the belief in the truth of results is based solely on the authority of one's own reason, and such that one is conscious of the power to apply these results appropriately and correctly. Within a narrow range, this independence may be attained very early; even the young pupil may have it, widening its scope as his attainments increase. Indeed, good teaching strives to cultivate this independence from the outset, and the teacher who is to cultivate it must himself evidently possess it and in a distinctly higher grade. The pupil, the teacher, the scholar, each should be in his degree master of his work. The teacher's mastery requires that he teach on his own authority, and not on that of others (though the mastery of the teacher need not be in the same degree productive as the mastery of the scholar). Unfortunately, our teachers have only too often not attained this independence. Too frequently a text is taught rather than the subject, the teacher having in reality not gone beyond the pupil's stage of mastery.

It is easier to point out the difficulties than to specify the remedy. The attainment of some measure of scientific independence suggests itself. This is the German plan, but it is not now feasible here. It seems to me, however, that somewhat can be done even within the limits of the college course. Our present methods and programmes tend in one respect to encourage diffidence rather than independence.

Mathematics requires to be digested and assimilated as well as acquired. The trend of our work is to cover as many subjects and as much ground as possible, and this continual advance with its successive generalizations and its revelations of incompleteness in what has been accepted as complete tends to cultivate a lack of confidence, accentuated if the student is embarrassed by difficulties of detail. This feeling of uncertainty takes shape in reliance upon others (book or person) to pronounce final judgment upon whatever is done. Perhaps something could be done to give the student the more secure grip on the secondary school subjects which the teacher must have by taking up these subjects again at the close of the college course and treating their principles and methods (not new details, as a rule), with the greater thoroughness and breadth of view which are made possible by the collegiate mathematics as outlined in the minimum course above.

VI. *Difference between the work of the prospective teacher and the prospective graduate student of mathematics.*—In the college this difference may be very slight.

VII. The teachers needed (in college) to carry out the above programme fully must have the wide horizon of a university training.

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ON A MEMOIR BY RICCARDO DE PAOLIS.

BY PROFESSOR CHARLOTTE ANGAS SCOTT.

ABOUT twenty years ago de Paolis published a series of memoirs* dealing with the (2. 1) transformation of the plane; of these the second and third are concerned with applications of the theory to non-euclidean geometry and to

* "Le trasformazioni piane doppie"; *Atti d. r. Accad. d. Lincei*, series 3ª, vol. 1 (1877); pp. 511–544.

"La trasformazione piana doppia di secondo ordine, e la sua applicazione alla geometria non euclidea"; *Atti d. r. Accad. d. Lincei*, series 3ª, vol. 2 (1878); pp. 31–50.

"La trasformazione piana doppia di terzo ordine, primo genere, e la sua applicazione alle curve del quarto ordine"; *Atti d. r. Accad. d. Lincei*, series 3ª, vol. 2 (1878); pp. 851–878.

quartic curves, while the first treats of the general theory of the (2, 1) transformation. De Paolis employs no analysis, all his results are obtained simply by geometrical considerations. The curves of the first plane (piano semplice) which correspond to the straight lines of the second plane (piano doppio) have only two free intersections, all the other common points being absorbed by the fixed (or fundamental) points by means of which the conditions to which the curves are subject are imposed. This, taking up a question which arises in connection with a memoir by Clebsch,* is the next in order to Cremona's memoirs,† the (2, 1) transformation naturally following the (1, 1). This very striking memoir, being already classical, is a fair subject for discussion; and in the following pages I propose to consider,‡ in particular, de Paolis's classification of fundamental points and lines, which in my opinion he has based on symptoms of differences instead of on the real causes. When the (2, 1) transformation is regarded as but one case in the general theory of ($\varkappa$, 1) transformation, some of the special properties—those, in fact, to which its peculiar interest is largely due—take on a rather different aspect.

The relation connecting the dimensions k of a system of curves of genus p with the number $\varkappa$ of variable intersections, is known to be

$$\varkappa - (k-1) = p,$$

if the fundamental points are entirely arbitrary in position, and consequently impose independent conditions;

$$\varkappa - (k-1) \leq p-1,$$

if the points are connected in such a way that the conditions imposed are not all independent. This shows that for the (2, 1) transformation with arbitrary fundamental points,

$$p + k = 3.$$

Hence, as shown by de Paolis, there are two cases to consider, namely,

$$p = 1,\ k = 2; \qquad p = 0,\ k = 3.$$

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*Clebsch, "Ueber den Zusammenhang einer Classe von Flächenabbildungen mit der Zweitheilung der Abel'schen Functionen"; *Math. Annalen*, vol. 3 (1871).

†Cremona, "Sulle trasformazioni geometriche delle figure piane"; *Memorie d. Accad. d. Bologna*, series 2ª, vol. 2 (1863); vol. 5 (1865).

‡I have not attempted to keep to the lettering of the memoir, preferring rather to use that adopted in other places.

In the first of these the transformation net is composed of all curves that satisfy the assigned conditions; in the second, these curves form a three-fold infinity, from which the net is selected by means of three determining curves. Now so far as the determination by fundamental points is concerned, these three curves may be any three independent ones of the system that have not any additional points in common. The same thing holds whatever value z may have; the system of curves determined by the fundamental points is of dimensions k, and to obtain a transformation net we virtually impose $k-2$ extrinsic conditions, these being implied in the choice of the three determining curves. The theory of linear systems contained in a system of the same order, but a greater number of dimensions, has received very little special attention, owing to the fact that the theorems relating to such a system are simply deducible from those for the complete system; but there are one or two points in the theory that deserve to be remarked, and they are the ones that seem to me to throw some light on the true nature of the differences that de Paolis has attempted to formulate in dividing fundamental points, as also fundamental curves, into three species.

Let the curves, φ, of the first plane that correspond to the straight lines of the second plane be given as of order σ, having fixed multiple points of orders $\rho_1, \rho_2, \cdots$, at the fundamental points. There may be also fundamental curves. A curve β, of order ω, with multiple points of orders $\tau_1, \tau_2 \cdots$ at the fundamental points, is called fundamental if it meets the curves of the given system only at the fixed points, that is, if

$$\sum \rho_i \tau_i = \sigma\omega.$$

These curves have been minutely studied by various Italian writers,* who have shown that any such curve is in general rational, and precisely determined by the fundamental points through which it passes, facts which lead to the equations

$$\sum \tau_i^2 = \omega^2 + 1, \quad \sum \tau_i = 3\omega - 1 \qquad \text{(Bertini)}.$$

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*Caporali, "Sopra i sistemi lineari triplamente infiniti di curve algebriche piane"; *Collect. Mat. In Mem. Dom. Chelini*, 1879.

Jung, "Sui sistemi lineari di curve algebriche di genere qualunque"; *Rend. del R. Ist. Lomb.*, series 2ª, vol. 20 (1887).

Bertini, "Sulle curve fondamentali dei sistemi lineari di curve piane algebriche"; *Rend. del Circ. Mat. di Palermo*, vol. 3 (1889).

Castelnuovo, "Ricerche generali sopra i sistemi lineari di curve piane"; *Mem. della R. Accad. di Sc. di Torino*, series 2ª, vol. 42 (1891).

If however the fundamental points are specialized in such a manner that the conditions actually imposed on the curves φ are fewer, by e, than the conditions apparently imposed (that is, if the fundamental points have an *excess* * e for the curves φ, or if the system of curves φ is of *sovrabbondanza* † e), then β need not be rational; its genus may have any value that does not exceed e (Castelnuovo). Any irreducible fundamental curve β is a factor in k independent curves of the system, that is, in a contained $(k-1)$-fold system; and there is the general theorem that β will be a *non-repeated* factor in the generic curve of this $(k-1)$-fold system.

As regards the fundamental points and curves, two questions call for investigation. The first relates to the possibility of a curve with a multiple point of order higher than the stated value ρ at a fundamental point, or, we may say, to the existence of a φ that shall be irregular at one or more of the fundamental points; the second to the possibility of a curve with β as a repeated factor, without any specialization in the position of the fundamental points. These two questions I have considered in another place; ‡ hence the results only need be mentioned here.

On the first question the conclusion is that the k-fold system of curves contains a curve that is irregular at any fundamental point whose order is less than k (or, by an extension of the argument, at any m fundamental points the sum of whose orders does not exceed $k-m$); but in selecting from the complete system three curves for the determination of a net, no two must be chosen that are irregular at the same point, for such a choice would introduce new fundamental points indefinitely near to the stated ones. Irregularity on one curve at any fundamental point does not of itself imply any specialization in the fundamental points.

As to the second question, the result is that the $k+1$ determining curves of the system can be chosen as the first $k+1$ of the following series:—

one, that does not contain β,

two, that contain β but neither contain nor produce β^2 (*i. e.* when taken in any linear combination),

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* Macaulay, "Point groups in relation to curves," *Proc. Lond. Math. Soc.*, vol. 26 (1895).

† Castelnuovo, loc. cit.

‡ "Note on linear systems of curves." *Nieuw Archief voor Wiskunde*, Tweede Reeks, Derde Deel, 1898; also BULLETIN, vol. 4 (1897–8); p. 88.

three, that contain β^2, but neither contain nor produce β^3,

.....

.....

$h+1$, that contain β^h, but neither contain nor produce β^{h+1}, where every $\rho \equiv h$ times the corresponding τ, and at least one $\rho < (h+1)$ times the corresponding τ.

If however $k+1 > \frac{1}{2}(h+1)(h+2)$, there are also curves, $k+1-\frac{1}{2}(h+1)(h+2)$ in number, containing higher powers of β; the numbers containing successive powers being N_{h+1}, N_{h+2}, etc., no general formula for an N can be given, all that can be said is that the successive N's do not increase. As these last contain β to a power so high that $h\tau > \rho$ at some one point, the curves are irregular. Hence it is convenient to distinguish the two parts of the series as regular and irregular. It is at once evident that any one of the curves can be modified by the addition of multiples of later members of the series without affecting its relation to β.

If $k = 2$, the curves by which the net is determined are necessarily the first three, that is,

$$\varphi_1, \quad \beta\psi_2, \quad \beta\psi_3.$$

If $k > 2$, any three may be chosen that have not any fixed points in common other than those expressly provided for at the fundamental points. One must obviously be φ_1; of the other two, one may belong to the irregular part of the series, but not both, inasmuch as this would cause irregularity at one point on both curves.

For example, the 12-fold system of 15-ics, with multiple points of orders 8, 6, 6, 5, 4, 4, 4 has the cubic 2, 1, 1, 1, 1, 1, 1 as a fundamental curve; and $h = 4$. The complete system can therefore be written as follows, where all thirteen curves are regular:

$$\varphi_1, \quad \beta(\psi_1, \psi_2), \quad \beta^2(\chi_1, \chi_2, \chi_3), \quad \beta^3(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4), \quad \beta^4(\zeta_1, \zeta_2, \zeta_3).$$

The system residual to β^4 is the net of cubics with fixed points of orders 0, 2, 2, 1, 0, 0, 0, which is simply the straight line $A_2A_3\times$ the net of conics through $A_2A_3A_4$. Hence in this case we must not choose, for the determination of a net, two curves containing β^4, even though both would be regular, for the net thus determined would have other fixed points, namely, the three points in which A_2A_3 meets the 15-ic φ_1. But with the 12-fold system of 14-ics with points of order 8, 6, 4, 4, 4, 4, 4, which has the same fundamental curve, there is not this restriction on freedom of choice.

We can at once write down a system with any possible fundamental curve, and with any arbitrary residual system, by means of the theorem :—If a curve β of order ω, rational and precisely determined by fixed points of order $\tau_1, \tau_2, \cdots$, which impose independent conditions, has h free intersections with the generic curve of a q-fold system of n-ics conditioned solely by its behavior at the fixed points, having these as multiple points of orders $r_1, r_2, \cdots$, then β is a fundamental curve of the system of $(n + h\omega)$-ics determined by multiple points of orders $r_1 + h\tau_1, r_2 + h\tau_2, \cdots$, at the fixed points.

The fundamental curve is not possible unless the numbers expressing the orders of its multiple points satisfy the relations

$$\tfrac{1}{2}\Sigma\tau(\tau+1) = \tfrac{1}{2}\omega(\omega+3) - \delta,$$

$$\tfrac{1}{2}\Sigma\tau(\tau-1) = \tfrac{1}{2}(\omega-1)\ (\omega-2),$$

and then δ simple points must be taken on β to complete its determination.

The proof of the theorem is obvious. The number of free intersections of β with the generic curve of the system is

$$\begin{aligned}\omega(n + h\omega) - \Sigma\tau(r + h\tau) &= n\omega - \Sigma r\tau + h\ (\omega^2 - \Sigma\tau^2)\\ &= h\ [1 + \omega^2 - \Sigma\tau^2] = 0.\end{aligned}$$

Hence β is either an h-fold factor or a fundamental curve ; but as the dimensions of the system

$$\geq \tfrac{1}{2}(n + h\omega)\ (n + h\omega + 3) - \tfrac{1}{2}\Sigma(r + h\tau)\ (r + 1 + h\tau)$$

$$\geq q + \tfrac{1}{2}h\ (h+1) > q,$$

it follows that the system of $(n + h\omega)$-ics is more extensive than the given system of n-ics, hence it is not simply this system with an extra factor β^h, but a more extensive system with β as a fundamental curve.

By means of this theorem we can write down a system with any desired fundamental curve, choosing a residual system at random. For example, if the fundamental curve be a rational cubic, we must use also for its complete determination six simple points on the curve, which can then be expresssed as $2,1^6$. If the residual system is to be *all* conics, then $h = 2 \times 3 = 6$; hence the desired system consists of 20-ics $(6 \times 3 + 2)$, with one multiple point of order 12, and six sextuple points.

In the general rational transformation, in which the coördinates of a point in the second plane are given rationally in terms of the coördinates of a point in the first plane by equations of the form

$$\varphi_1 : \varphi_2 : \varphi_3 = x_1' : x_2' : x_3',$$

to a single point P in the first plane there corresponds a single point P' in the second plane; but to this one point in the second plane there correspond in the first plane all the κ points $P_1, P_2, \cdots, P_\kappa$ which are the intersections of the pencil of φ's through P. If however P is a fundamental point of order ρ, *e. g.*, the point (0, 0, 1), there is no longer any unique correspondent P'. The φ's being arranged by powers of x_3, the equations become

$$\frac{x_1'}{u_\rho x_3^{\sigma-\rho} + u_{\rho+1} x_3^{\sigma-\rho-1} + \cdots} = \frac{x_2'}{v_\rho x_3^{\sigma-\rho} + \ldots} = \frac{x_3'}{w_\rho x_3^{\sigma-\rho} + \cdots};$$

the correspondent to the actual point (0, 0, 1) is absolutely indeterminate, for all these denominators are zero; but taking a point indefinitely near to (0, 0, 1), in a definite direction given by $x_1 = \lambda x_2$, the corresponding point is determinate, the equations now becoming

$$\frac{x_1'}{u_\rho(\lambda, 1)} = \frac{x_2'}{v_\rho(\lambda, 1)} = \frac{x_3'}{w_\rho(\lambda, 1)}.$$

Taking in succession points close to (0, 0, 1) in all directions, that is, allowing P to describe an indefinitely small simple circuit about (0, 0, 1), P' describes a curve α', rational and of order ρ, inasmuch as the x_1', x_2', x_3' are rationally expressed in terms of the parameter λ by these equations. (See Clebsch-Lindemann, and other places.) Returning to the first plane, the complete correspondent to α_i' is a curve α_i of order $\sigma\rho_i$, which can be shown* to have at A_j a multiple point of order $\rho_i\rho_j$, but at A_i one of order $\rho_i^2 + 1$. This curve α_i is the complement of the fundamental point, the locus of the $\kappa - 1$ points associated with a point that describes a circuit about A_i. If $\rho_i = 1$, it is simply the particular φ that has a double point at A_i.

Thus while any ordinary point of the first plane corresponds to a point of the second plane, and is associated with $\kappa - 1$ points of the first plane, there are exceptional points, namely, the fundamental points; any one of these in

*C. A. Scott, "Studies in the transformation of plane algebraic curves"; *Quar. Jour. of Math.*, vol. 29 (1898).

general corresponds to a curve, and is associated with a curve which replaces the $z-1$ points.

If however one φ, φ_s suppose, be irregular at (0, 0, 1), the lowest terms being of degree $\rho' > \rho$, the equations are

$$\frac{x_1'}{u_\rho + \cdots} = \frac{x_2'}{v_\rho + \cdots} = \frac{x_3'}{w_{\rho'} + \cdots}.$$

As x_1, x_2 are indefinitely small, and $\rho' > \rho$, to a point in a definite direction from (0, 0, 1) there corresponds a definite point on $x_3' = 0$; and as the complete correspondent to this is $\varphi_s = 0$, the curve complementary to A_i is φ_s. For consistency of algebraic statement, it appears that the fundamental curve in the second plane must be taken as $x_3'^{\rho}$, and the curve complementary to A_i as φ_s^{ρ}.

If φ_s is irregular at another fundamental point of stated order ρ_k, the correspondent to this is the same line x_3' taken ρ_k times, and the complement is the same φ_s taken ρ_k times.

The existence of the irregular φ affects the position of the points associated with any point P in the immediate neighborhood of A_i. These are to be determined as the free intersections of two φ's through P; two such φ's have in general ρ^2 fixed intersections at A_i, but if one of the two has a ρ'-point, the number is $\rho\rho'$; hence of the usually free intersections $\rho(\rho' - \rho)$ lie at A_i instead of only one. This number, $\rho(\rho' - \rho)$, is greater than unity unless $\rho = 1$, $\rho' = 2$. Now the raising of a simple point to a double point involves the imposition of two conditions only, hence is always possible in a net; it seems better therefore not to regard this case as an irregularity, seeing that it is the necessary occurrence.

If moreover φ_s is irregular also at A_j, then, taking P close to A_i as before, instead of the usual ρ_j^2 intersections at A_j there are $\rho_j\rho_j'$; thus of the z points of the group, $\rho_j(\rho_j' - \rho_j)$ lie at A_j. It is immaterial in what direction P lies from A_i, for an irregular φ belongs to the pencil of φ's determined by any direction whatever.

Hence irregularity on a φ will interfere with the customary association of a fundamental point with a curve; the fundamental point may have complementary points lying at other fundamental points, or even coinciding with itself. In the cases treated by de Paolis, since $z = 2$ there is only one point associated with A_i, and this point

(i) describes a curve; A_i is then a fundamental point of the first species;

(ii) lies at another fundamental point A_j; A_i is then a fundamental point of the second species;

(iii) lies at A_i, which is then a fundamental point of the third species. Such a point arises, as has just been shown, from irregularity on one φ; it can arise also from another cause, the presence of what de Paolis calls a double point, to be explained later.

For a non-specialized transformation system with $\varkappa = 2$, $p + k = 3$. If $p = 1$, $k = 2$; the only "irregularity" possible on a φ is the existence of a double point where the stated conditions impose only a simple point, and it has already been pointed out that this cannot properly be regarded as an irregularity. If $p = 0$, $k = 3$; we cannot produce irregularity on one φ at two points, for this requires the imposition of four conditions, and is therefore possible in a system only if the generic curve has as many as four degrees of freedom. Hence the fundamental points of the second species do not present themselves unless the determining points of the system are specialized, in which case $p + k > 3$. The fundamental points of the third species can occur, for $k = 3$ allows us to raise a double point to a triple point, and then an extra intersection lies at A_i, A_i is associated with itself; there is no complementary curve, for the two points of a group are fully accounted for.

This classification of fundamental points lends itself to the case $\varkappa = 2$, the characteristics of the three species being then mutually exclusive, inasmuch as the one point associated with any point P, near to A_i, cannot satisfy more than one of the three conditions. But if $\varkappa > 2$, *e. g.*, $= 4$, and k is sufficiently great, we may have the three points complementary to P exhibiting all three characteristics. As P describes an indefinitely small simple circuit about A_i, P_1 may describe a curve, P_2 may lie at A_i, P_3 at A_i, and then the curve described by P_1 will be simply the irregular φ, φ_3. Thus de Paolis' classification is based on properties which are insufficient for purposes of discrimination when $\varkappa > 2$.

Two points of a general $\varkappa$ group (the intersections of a pencil of φ's), can come together only if the φ's have contact; the locus of such points is therefore the Jacobian of φ_1, φ_2, φ_3. On forming this, in certain circumstances factors will present themselves, possibly to be rejected in finding the locus of points of true contact. De Paolis calls this locus the "curva doppia"; it is the Jacobian reduced by the rejection of irrelevant factors. As a point P_1, together with its associate P_2, describes the Jacobian, the correspondent P' describes a curve in the second plane, the "curva limita" of de Paolis [synoptic curve, *Q. J.*, loc. cit.]. In the general theory, with $\varkappa > 2$, we have to take into ac-

count also the co-jacobian, the locus of the remaining points $P_3, P_4, \cdots, P_\varkappa$; a detailed study of the relations of these curves, and their use in transformation, is to be found in the article in the *Quarterly Journal* already referred to.

Examining the equations of the Jacobian, it is seen to represent a curve of order 3 $(\sigma - 1)$, with multiple points of order $3\rho - 1$. If however one φ is irregular at A_i, having there a multiple point of order ρ', the order of this point on the Jacobian is $\rho' + 2\rho - 2$; the terms of this order are affected with a numerical factor $\rho' - \rho$. If now $\rho' = \rho + 1$, this has the normal value $3\rho - 1$; but if $\rho' \geq \rho + 2$, then $\rho' + 2\rho - 2 \geq 3\rho$. Thus the Jacobian is not affected by the simplest type of irregularity, that in which the order of a fundamental ρ-point is augmented by unity on one φ. With the restriction on the scope of the investigation adopted by de Paolis, namely, $\varkappa = 2$, it is not possible to have $\rho' = \rho + 2$, hence this irregularity on the Jacobian is not mentioned. The irregularity that is considered on p. 523 has a different origin; it is due to the presence of a double point, whose nature is considered separately.

Corresponding to the fundamental points in the first plane there are fundamental curves in the second plane, which de Paolis classes as of the first, second, or third kind, according to the nature of the fundamental points from which they are derived. He makes the statement, as regards those of the second and third kind, that they meet the curves of the second plane which correspond to the straight lines of the first plane in fixed points only. This is not the case if $\varkappa > 2$; but possibly it is true for $\varkappa = 2$, as stated by de Paolis, and not only for the curves that correspond to straight lines, but also for the correspondents to any curves in the first plane, though I must confess I find de Paolis' argument somewhat obscure.

If there be in the first plane a fundamental curve β, the equations of transformation are

$$\varphi_1 : \beta^{h_2}\varphi'_2 : \beta^{h_3}\varphi'_3 = x_1' : x_2' : x_3'.$$

Corresponding to any point on β we have in the second plane the point $B'(1, 0, 0)$; to the point B' there correspond all the intersections of $\beta^{h_2}\varphi'_2$, $\beta^{h_3}\varphi'_3$ that do not lie at the fundamental points, $\varkappa - h_2h_3$ in number. Hence the fundamental curve β replaces h_2h_3 of a group of $\varkappa$ points, thus displaying another exception to the general laws of the transformation. Of course h_2h_3 is necessarily less than $\varkappa$. If $h_2 = h_3 = h$, to any point in the immediate neighbor-

hood of B', in the direction $x_2' = \lambda x_3'$, there correspond h points on β, inasmuch as $\beta = 0$ meets $\psi_2 - \lambda\psi_3 = 0$ in precisely h points; thus if $h = 1$, that is, if β is a fundamental curve of the simplest type, as a point P' describes an indefinitely small circuit about B' one of the corresponding points describes the curve β; the relation of B' to β is precisely analogous to the relation of A to a'. This however does not hold if h is greater than unity.

If $h_2 < h_3$, to any point in the neighborhood of B' there correspond h_2 fixed points on β, determined by $\psi_2 = 0$.

Thus if $h_2 h_3 > 1$, one or more of the correspondents to β is β itself, the remainder being a set of $\varkappa - h_2 h_3$ points.

If however ψ_2, ψ_3 have a common factor γ, we have in the simplest case the equations

$$\varphi_1 : \beta\gamma\vartheta_2 : \beta\gamma\vartheta_3 = x_1' : x_2' : x_3'.$$

To the point B' there now corresponds the reducible curve $\beta\gamma$, which is "monovalent" as regards the system of φ's; that is to say, if the φ is made to pass through one point on either component, it contains $\beta\gamma$ as a factor, the whole curve $\beta\gamma$ is equivalent to a single point as means of imposing conditions on the generic curve of the system of φ's. (Castelnuovo, loc. cit.) With any point on β there are associated the $\varkappa - 2$ intersections* of ϑ_2, ϑ_3, and the curve γ; this curve is therefore a part of the complement of β.

If finally the equations of transformation are

$$\varphi_1 : \beta^{k_2}\gamma^{k_2}\vartheta_2 : \beta^{k_3}\gamma^{k_3}\vartheta_3 = x_1' : x_2' : x_3',$$

then a group of $\varkappa$ associated elements, of which β forms a part, contains in the simplest case $\varkappa - h_2 h_3 - k_2 k_3$ points, not on $\beta\gamma$, together with β, $h_2 h_3$ times, and γ, $k_2 k_3$ times. If for instance $\varkappa = 4$, and the transformation net is

$$\varphi_1, \quad \beta\gamma\vartheta_2, \quad \beta^2\gamma\vartheta_3,$$

then a set of four associated elements one of which is β consists of β, twice, γ, once, and one point. De Paolis divides the fundamental curves in the first plane into three kinds, according as the single associate (the only one if $\varkappa = 2$) is a point, another curve, or the curve itself; but here again, just as in the case of the fundamental points, if $\varkappa > 2$ the characteristics may coexist in any combination, hence the classification cannot profitably be placed on this basis in general.

* There are $\varkappa - 2$ in the simplest case that can arise, but they may be fewer in number.

On p. 518 de Paolis asserts that while fundamental curves of the first and second species are rational, those of the third species are *hyperelliptic*. Surely he has here been misled by the (2, 1) correspondence of the points of such a a curve with directions proceeding from B'. Any fundamental curve β, of whatever species, is subject to the conditions formulated in the memoirs of Bertini and Castelnuovo, conditions which are a direct consequence of the fact that all the intersections of β with the generic φ are fixed. If the complete system from which the φ-net is derived by selection has no "sovrabbondanza," then β is rational, since its genus cannot exceed the sovrabbondanza of the system. For a specialized choice of fundamental points, a fundamental curve of any species may cease to be rational.

From the ordinary determinant expression for the Jacobian it is at once seen that a fundamental curve that is an (h_2, h_3) factor presents itself as an $(h_2 + h_3 - 1)$-fold factor in the Jacobian; rejecting it, the residual part, J_0, is de Paolis' "curva doppia." This cuts β in $h_2 + h_3 - 2$ points; if $h_2 = h_3 = h$, these $2(h-1)$ points are $h-1$ points of simple contact. These are the general facts; de Paolis naturally notices only the cases that can occur for $\varkappa = 2$. If β is a simple fundamental curve, then β is a simple factor to be rejected from the Jacobian, and the reduced Jacobian does not meet β; if however β is a fundamental curve of the third species ($h_2 = 1$, $h_3 = 2$), the factor β^2 is to be rejected, and the reduced Jacobian meets β in one point.

A point on the "curva doppia" is a point of the plane where two complementary points come together, that is, are consecutive; the φ's through such a point have simple contact, the common tangent being the line determined by these consecutive points, and one φ has a double point. If there is an actual *coincidence* of two complementary points, de Paolis characterizes it as a double point of the plane, stating that the continuous system of points of the first plane may contain a discrete system of points associated with themselves. Any two φ's through such a point have two intersections, not due to contact. If the point is not fundamental, and $\varkappa = 2$, this can only be by means of a common factor, which is consequently a fundamental curve. The net is then determined by

$$\varphi_1, \quad \beta\psi_2, \quad \beta\psi_3,$$

but as ψ_2, ψ_3 are to have a common point on β, which fixes the sole (hitherto) free intersection of ψ_2 and ψ_3, it follows

that ψ_2 is of the form $\psi_2 + \beta\vartheta_2$; hence the net is determined by

$$\varphi_1, \quad \beta\psi_2, \quad \beta^2\vartheta_3,$$

and the fundamental curve β is therefore of the third species, as de Paolis shows in a different manner (p. 524). The double point is the one point in which β and ψ_2 intersect; and as any φ that contains β is of the form $\beta\psi_2 + \lambda\beta^2\vartheta_3$, that is, $\beta^2(\psi_2 + \lambda\beta\vartheta_3)$, every such φ has necessarily a double point at this point.

De Paolis takes the view that all such double points are isolated from the Jacobian, because he found them on the fundamental curves, and these he rejected in forming the "curva doppia." If however $z > 2$, all the φ's through a point may have a double point there, without degeneration, that is, without the assistance of fundamental curves; this fact is outside de Paolis' argument, but it is essential to a full understanding. In this case the reduced Jacobian, the "curva doppia," has a double point. Such a point exists, for instance,* at (1, 0, 0) for the system φ_1, φ_2, φ_3, where

$$\varphi_1 = x_1(x_1 + x_2 + x_3)^2,$$
$$\varphi_2 = 2x_1x_2^2 + (x_2 - x_3)(x_2 + x_3)^2,$$
$$\varphi_3 = 2x_1x_3^2 - (x_2 - x_3)(x_2 + x_3)^2,$$

whose reduced Jacobian is

$$x_1(x_2 + x_3)^2 - (x_1 + x_2 + x_3)(x_2 - x_3)^2 = 0,$$

that is,

$$4x_1x_2x_3 - (x_2 + x_3)(x_2 - x_3)^2 = 0.$$

This point is of the same nature as the double point of which de Paolis speaks, but of higher order.

These are the only two ways in which all φ's of a pencil can have more than one intersection at a point that is not fundamental, independently of contact; either any two, and therefore all, have a common factor; or any two, and therefore all, have a multiple point.

The question whether a fundamental curve, which is a repeated factor in one or more φ's, shall or shall not be counted as a part of the reduced Jacobian, is after all a mere matter of definition; we may adopt any convention. A repeated factor (h-fold) in one φ, that is not a fundamental curve, presents itself as an $(h-1)$-fold factor in the

* *Quar. Jour. of Math.*, vol. 29 (1898), pp. 357 and 379.

Jacobian, and has to be counted as an integral part of this curve; whereas a simple factor has no special relation to it. This lends some support to the view, to which I incline, that a fundamental curve which occurs as a repeated factor ought to be looked upon as a part of the true Jacobian; and bearing in mind the fact just proved as regards multiple points of the plane, it appears that this convention gives the most consistent set of results.

Again, a double point may be at a fundamental point A; a point complementary to A is then at A, which is therefore a fundamental point of the third species. This is the second manner of specializing the fundamental point, referred to above. It is the only one that de Paolis considers in any detail; and his purely geometrical treatment, while it has led him to a correct formulation of the properties of such a point, is not very conclusive as to its nature, which however is extremely interesting. This is considered analytically in the *Quarterly Journal of Mathematics*;* as the results of the investigation throw light on a rather obscure part of the memoir of de Paolis, it appears suitable to give here a résumé.

The pencil formed by the φ's that proceed in any given direction from a fundamental point of order ρ, contains one special φ, with a cuspidal branch whose tangent lies in that direction. If the direction is one of the $3\rho - 1$ indicated by the tangents to the Jacobian, the special φ has not a simple cuspidal branch, but a branch with a higher singularity, met by its own tangent in four points. In the simplest case, this can be resolved into two simple branches in contact, and the φ might be referred to as tacnodal; but to allow for the more complicated singularity, the term *extranodal* seems more convenient, as it is applicable to all cases. (Analytically, the repeated factor in the terms of degree ρ is a factor in the terms of degree $\rho + 1$.) These extranodal φ's are determined by the $3\rho - 1$ tangents to the Jacobian; the fundamental point may, however, be specialized in such a manner that for *every* direction the special φ is extranodal instead of cuspidal. This happens necessarily for all directions if it happens for some one direction in addition to the $3\rho - 1$; and the Jacobian has then a point of order 3ρ, while the synoptic curve contains a' as a factor. The number of intersections at A of two φ's proceeding in any assigned direction AP is in general $\rho^2 + 1$, but in this case it is $\rho^2 + 2$; of these ρ^2 are the fixed intersections, which

* In the continuation of the article, already referred to, appearing in the current volume of the *Quar. Jour. of Math.*

are not enumerated in any $\varkappa$-group; thus two points of a $\varkappa$-group fall at P, in whatever direction P is taken. This may perhaps be most simply expressed by saying that the complement to A contains an evanescent circuit about A. This is what de Paolis refers to as a "punto fondamentale doppio"; it is one variety of the fundamental point of the third species.

The fundamental points and lines, the double points, and the Jacobian, in the first plane, with their correspondents in the second plane, form a kind of framework for the transformation, indicating everything exceptional; and a complete knowledge of them is essential to a full comprehension of the possible effects of the general rational transformation. If $\varkappa > 2$, the co-jacobian has to be taken into account also, and as I have here shown, the different species of fundamental points and lines can no longer be discriminated; hence the purely geometrical treatment becomes very involved, and it appears necessary to resort to analysis. Thus it is hardly to be expected that the treatment of transformation can be carried any further on the lines of Cremona and de Paolis; but the intrinsic interest of de Paolis' work is surely excuse enough for devoting some little space to it in a periodical of a critical and historical nature.

BRYN MAWR COLLEGE,
May, 1900.

NOTES.

THE Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will meet in the Ryerson physical laboratory of the University of Chicago, on Thursday and Friday, December 27th and 28th next. Titles and abstracts of papers to be read at this meeting should be in the hands of the Secretary of the Section not later than December 5th.

THE third (July) number of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Wave propagation over non-uniform conductors," by M. I. PUPIN; "Ueber Systeme von Differentialgleichungen denen vierfach periodische Functionen Genüge leisten," by M. KRAUSE; "On linear criteria for the determination of the radius of convergence of a power series," by E. B. VAN VLECK; "On the existence of the Green's function for the

most general simply connected plane region," by W. F. Osgood; "'D' lines on quadrics," by A. Pell; "Sundry metric theorems concerning n lines in a plane," by F. H. Loud; "An application of group theory to hydrodynamics," by E. J. Wilczynski; "Determination of an abstract simple group of order $2^7.3^6.5.7$ holoedrically isomorphic with a certain orthogonal group and with a certain hyperabelian group," by L. E. Dickson.

In the October (closing) number of volume I. of the *Transactions* it is proposed, for the convenience of the readers, to collect under an appropriate heading brief notes concerning the memoirs of the volume. For this purpose the contributors are requested to furnish a complete list of errata in their memoirs as published, and any further notes, for instance in the way of reference.

The German mathematical society held its annual meet- at Aachen, September 16th–23d, 1900, with Professor D. Hilbert as president, and Professor A. Gutzmer as secretary. The preliminary programme of papers to be presented is as follows:

(1) W. Ebert, Paris: "On the direct measurement of the motion of the earth's axis of rotation by means of stars near the pole."

(2) E. Jürgens, Aachen: "Computation of determinants."

(3) F. Klein, Göttingen: "Mechanics in the Encyclopædia of the mathematical sciences."

(4) A. Kneser, Dorpat: "On the development and present state of the calculus of variations."

(5) E. Kötter, Aachen: Subject to be announced.

(6) A. Marcuse, Berlin: "New development of the theory and practice of the determination of geographical and nautical positions."

(7) A. Marcuse, Berlin: "A new photographic universal instrument for the determination of geographic-astronomical positions."

(8) F. Meyer, Koenigsberg: "On singular binary forms and relations between sub-determinants."

(9) F. Meyer, Koenigsberg: "On geometrical theorems of the nature of those of Pascal and Desargues."

(10) H. Minkowski, Zürich: "On the concepts: length, surface, and volume."

(11) P. H. Schoute, Groningen: "On the number of points, right lines, planes, etc., of space of n dimensions."

(12) H. Schubert, Hamburg: "Zur Schlick'schen Ausgleichung der Schiffsvibrationen."

(13) D. SINTZOW, Ekaterinoslaw: "On connexes in space."

(14) P. STÄCKEL, Kiel: "On the theory of geodetic lines."

(15) A. WANGERIN, Halle: "Proof of a theorem relative to lines of curvature."

(16) A. WANGERIN, Halle: "Determination of all surfaces having constant curvature."

GAUTHIER-VILLARS announces the recent publication of Leçons sur la théorie des formes et la géométrie analytique supérieure, volume I, by H. ANDOYER; Recueil de problèmes de géométrie analytique, by FRANCOIS MICHEL; Analyse infinitésimale a l'usage des ingénieurs, volume I, by E. ROUCHÉ and L. LÉVY. The same publishing house has in preparation a translation into French of the Encyclopædia der mathematischen Wissenschaften, the first three volumes of which are in the editorial charge of JULES MOLK. The first volume to appear in print is volume II of the work which is expected from the press early in 1901. Among other works in preparation are a Cours d'analyse mathématique, by E. GOURSAT and an Essai sur les fondements de géométrie by W. B. RUSSELL. The second volumes of the above-named works of ANDOYER and of ROUCHÉ and LÉVY are in press.

THE Macmillan Company have in preparation an Introduction to celestial mechanics by Dr. F. R. MOULTON, associate professor of astronomy in the University of Chicago, and The principles of mechanics, by Professor FREDERICK SLATE, of the University of California.

THE several universities below offer during the winter semester 1900–1901 courses in mathematics as follows:

UNIVERSITY OF BERLIN.—By Professor L. FUCHS: Theory of elliptic functions, four hours; Introduction to the theory of differential equations, four hours; Seminar, two hours.—By Professor H. A. SCHWARZ: Differential calculus, four hours; Applications of the theory of elliptic functions, two hours; Calculus of variations, four hours; Colloquium, two hours; Seminar, two hours.—By Professor G. FROBENIUS: Theory of algebraic equations, four hours; Seminar, two hours.—By Professor J. KNOBLAUCH: Analytical geometry, four hours; Definite integrals, four hours; Exercises, one hour.—By Professor K. HENSEL; Integral cal-

culus, four hours; Theory of numbers, four hours; Colloquium, two hours.—By Professor R. LEHMANN-FILHÉS: Analytical mechanics, four hours; Exercises, one hour.

UNIVERSITY OF BONN.—By Professor L. HEFFTER: Analytical geometry, four hours; Elliptic functions, four hours; Exercises in geometry, one hour.—By Professor H. KORTUM: Infinitesimal calculus II, four hours; Curved lines and surfaces, two hours; Mathematical seminar, two hours.—By Professor R. LIPSCHITZ: Analytical mechanics, four hours; Mathematical seminar, two hours.

UNIVERSITY OF BRESLAU.—By Professor J. ROSANES: Algebraic equations, four hours; Elements of the theory of differential equations, two hours; Seminar, one hour.—By Professor R. STURM: Infinitesimal calculus, four hours; Theory of numbers, two hours; Seminar, one hour.—By Dr. F. LONDON: Analytical mechanics, four hours, with exercises, one hour.

UNIVERSITY OF ERLANGEN.—By Professor M. NOETHER: Infinitesimal calculus, four hours; Introduction to algebraic analysis and the theory of functions, four hours; Mathematical exercises.—By Professor P. GORDAN: Analytical Geometry, four hours; Algebra, four hours; Seminar.

UNIVERSITY OF FREIBURG.—By Professor J. LÜROTH: Analytical mechanics, five hours; Trigonometry, three hours; Seminar, one hour.—By Professor L. STICKELBERGER: Plane analytical geometry, four hours; Theory of numbers, three hours.—By Dr. A. LOEWY: Differential calculus, three hours; Theory of surfaces, three hours; By Dr. E. REBMANN: Methodology of mathematical instruction, two hours.

UNIVERSITY OF GIESSEN.—By Professor M. PASCH: Infinitesimal calculus, four hours; Selected chapters of analytical geometry, two hours; Exercises in the elements of higher mathematics, one hour; Seminar, one hour.—By Professor E. NETTO: Analytical geometry of space, four hours; Algebraic equations, two hours; Seminar, one hour.—By Professor R. HAUSSNER: Introduction to the theory of differential equations, four hours; Technical mechanics, four hours; Theory of determinants, two hours.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Projective geometry, four hours, with exercises, two hours.

—By Professor D. HILBERT: Theory of functions, four hours; Partial differential equations, four hours, with seminar in the latter, two hours.—By Professor M. BRENDEL: Rotational motion of the heavenly bodies, two hours; Exercises in numerical reckoning, two hours.—By Professor E. WIECHERT: Analytical mechanics, four hours.—By Professor F. SCHILLING: Integral calculus, four hours, with exercises, one hour.—By Professor AMBRONN: Method of least squares, two hours.—By Professor K. BOHLMANN: Mathematical principles of insurance, three hours; Mathematical statistics, two hours; Seminar in insurance, two hours.—By Dr. E. ZERMELO: Mengenlehre, two hours; Kinetic theory of gases, two hours.—By Dr. J. SOMMER: Theory of algebraic functions, two hours.

UNIVERSITY OF GREIFSWALD.—By Professor W. THOMÉ: Mechanics I, four hours; Algebra, four hours; Seminar, two hours.—By Professor E. STUDY: Infinitesimal calculus II, four hours; Theory of functions II, four hours; Seminar, two hours.

UNIVERSITY OF HALLE.—By Professor G. CANTOR: Infinitesimal calculus, five hours.—By Professor A. WANGERIN: Chapters in higher geodesy, one hour; Integral calculus with exercises, four hours; Applications of elliptic functions, two hours; Calculus of variations, two hours.—By Professor V. EBERHARD: Theory of algebraic equations, four hours; Numerical equations with exercises, one hour.—By Dr. H. GRASSMANN: Ordinary differential equations, two hours; Exercises in descriptive geometry, one hour.—By Dr. E. NEUMANN: Theory of continued fractions, two hours.

UNIVERSITY OF HEIDELBERG.—By Professor L. KOENIGSBERGER: Higher algebra, four hours; Elements of the theory of differential equations, two hours; Theory of numbers, two hours; Seminar, two hours.—By Professor H. VALENTINER; Method of least squares, two hours.—By Professor M. CANTOR: Infinitesimal calculus, four hours, with exercises, one hour; Political arithmetic, two hours.—By Professor F. EISENLOHR: Theoretical optics, four hours; Infinitesimal calculus, five hours; Potential, two hours.—By Professor K. KOEHLER: Synthetic geometry of space, three hours.—By Professor G. LANDSBERG: Descriptive geometry, four hours; Selected chapters of theory of functions, continued from preceding semester, two hours.—By Dr. K. BOEHM: The principles of Heinrich Hertz's

mechanics, two hours; Gauss's memoir on the theory of surfaces, one hour.

UNIVERSITY OF INNSBRUCK.—By Professor O. STOLZ: Real differential and integral calculus, four hours; Theory of functions of a complex variable after Cauchy and Weierstrass, three hours.—By Professor W. WIRTINGER: Linear partial differential equations, three hours; Eulerian integrals, two hours; Mathematical seminar, two hours.—By Dr. K. ZINDLER: Descriptive geometry with exercises, four hours; Application of the differential calculus to geometry, two hours.

UNIVERSITY OF JENA.—By Professor J. THOMAE: Application of infinitesimal calculus to geometry, four hours; Definite integrals and hypergeometric series, two hours; Seminar, two hours.—By Professor A. GUTZMER: Integral calculus, four hours, with seminar exercises, one hour; Elements of descriptive geometry with exercises, four hours.—By Professor G. FREGE: Analytical geometry of space, four hours; Concepts, one hour.

UNIVERSITY OF KIEL.—By Professor L. POCHHAMMER: Application of infinitesimal calculus to geometry, three hours; Introduction to the theory of functions, three hours; Seminar, two hours.—By Professor P. STÄCKEL: Higher analysis II, four hours; Stereometric drawing, four hours; Calculus of variations, one hour; Seminar, one hour.

UNIVERSITY OF KOENIGSBERG.—By Professor F. MEYER: Selected chapters of higher geometry, two hours; Integral calculus, three hours, with exercises, one hour; Seminar, one hour and a half.—By Professor A. SCHOENFLIES: Mechanics, four hours.—By Professor L. SAALSCHÜTZ: Theory of calculus of finite differences, two hours; Introduction to algebraic analysis, three hours, with exercises, one hour.—By Dr. E. MÜLLER: Analytical geometry of space, two hours; Perspective, two hours.

UNIVERSITY OF LEIPSIC.—Professor SCHEIBNER offers no courses this semester, and those of Professor MAYER will be announced later.—By Professor C. NEUMANN: Application of the infinitesimal calculus to geometry, four hours; Seminar, one hour.—By Professor O. HÖLDER: Infinitesimal calculus, five hours; Application of elliptic functions, one hour; Mathematical seminar, one hour.—By Professor L. BOLTZMANN: Analytical mechanics, five hours.—By Professor F. ENGEL: Projective geometry and homogeneous co-

ordinates, three hours; Differential invariants, two hours; Seminar, one hour.—By Dr. F. HAUSDORFF: Theory of probability, three hours; Map projection, two hours.—By Dr. G. KOWALWOSKI: Theory and application of determinants, two hours; Theory of partial differential equations, two hours; Seminar in differential invariants, in conjunction with Professor Engel, one hour.—By Dr. H. LIEBMANN: Elementary theory of numbers, two hours; Foundations of geometry, two hours.

UNIVERSITY OF MARBURG.—By Professor F. SCHOTTKY: Elliptic functions, four hours; Algebraic analysis, three hours; Seminar, two hours.—By Professor E. HESS: Integral calculus, five hours; Selected chapters of geometry, three hours; Spherical trigonometry, two hours; Seminar, two hours.—By Dr. F. v. DALGWIK: Analytical geometry of the plane and space, five hours, with exercises, one hour; Descriptive geometry, with exercises, two hours.

UNIVERSITY OF MUNICH.—By Professor G. BAUER: Algebra; subject of second course to be announced; Seminar.—By Professor F. LINDEMANN: Plane analytical geometry; Introduction to ordinary and partial differential equations; Mathematical principles of life insurance; Seminar.—By Professor A. PRINGSHEIM: Differential calculus, with exercises; Elliptic functions.—By Dr. K. DÖHLEMANN: Descriptive geometry, with exercises; Kinematics.—By Dr. E. v. WEBER: Introduction to analysis; Determinants with applications; Selected chapters of elementary geometry.—By Dr. J. GÖTTLER: Theory and application of the functions of Lamé, Bessel and Laplace.

THE STEINER PRIZE.—The Academy of sciences of Berlin at the Leibniz Meeting, July 4, 1895, announced the following problem for the Steiner Prize: "To completely solve any important, hitherto unsolved problem relating to the theory of curved surfaces, taking into account, so far as possible, the methods and principles evolved by Steiner. It is required that sufficient analytical explanations shall accompany the geometrical investigations to verify the correctness and completeness of the solution. Without wishing to limit the choice of subject, the Academy takes the opportunity to call attention to the special problems to which Steiner has referred in his general remark at the end of his second paper on maximum and minimum in figures in a plane, on a sphere, and in space."

No paper on this subject was received. In accordance

with the terms of the Steiner foundation, the Academy has utilized the prize of 6,000 marks, thus unawarded, for the purpose of recognizing certain important geometrical contributions published during the last few years. One third of it was awarded to each of the following: To DR. KARL FRIEDRICH GEISER, professor at the polytechnic school at Zurich, for his individual researches in geometry and his services in the publication of a part of Steiner's lectures; to DR. DAVID HILBERT, professor at the University of Göttingen, for his important researches on the axioms of geometry and for the advancement which analytic geometry has experienced from his work on the theory of invariants; to DR. FERDINAND LINDEMANN, professor at the University of Munich, who has earned special distinction in geometry by his celebrated discussion of the quadrature of the circle, as well as by editing Clebsch's Vorlesungen über Geometrie.

For the year 1905 the Academy again announces the foregoing problem, which has remained unsolved. For its solution a prize of 4,000 marks is offered, with an additional sum of 2,000 marks. Papers offered in competition may be written in German, French, English, Italian or Latin, and must be submitted before December 31, 1904. The result will be announced at the Leibniz meeting of 1905. No manuscript revealing the name of the author will be accepted. Each manuscript is to bear a mark or nom-de-plume, and to be accompanied by a sealed envelope containing the name and address of the author and bearing outside the corresponding mark or assumed name. Manuscripts should be sent to the Bureau of the Academy, Berlin NW. 7, Universitäts-strasse, 8.

FROM an article in *Science*, August 31, 1900, it appears that the degree of doctor of philosophy was conferred by American universities during the past year on 233 candidates, of whom 11 presented dissertations in mathematics. The list of mathematical doctorates, with titles of theses and names of universities conferring the degrees, is as follows: L. P. EISENHART, Johns Hopkins University, "Infinitesimal deformation of surface"; W. GILLESPIE, University of Chicago, "Determination of all hyperelliptic integrals of the first kind of genus 3 reducible to elliptic integrals by transformations of the second and third degrees"; H. E. HAWKES, Yale University, "Examination and extension of Peirce's linear associative algebra"; D. N. LEHMER, University of Chicago, "Asymptotic evaluation of certain totient sums"; J. H. MCDONALD, University of Chicago,

"Concerning the system of the binary cubic and quadratic with application to the reduction of hyperelliptic integrals to elliptic integrals by a transformation of order four"; C. R. McINNES, Johns Hopkins University, "Superosculated sections of surfaces"; H. C. MORENO, Clark University, "On ruled loci in n-fold space"; F. R. MOULTON, University of Chicago, "A particular class of periodic solutions of the problem of three bodies"; S. E. SLOCUM, Clark University, "On the continuity of groups generated by infinitesimal transformations"; F. B. WILLIAMS, Clark University, "Geometry on ruled quartic surfaces"; A. D. YOCUM, University of Pennsylvania. "An inquiry into the fundamental processes of addition and subtraction."

THE Hopkins prize of Cambridge University for the period 1894–1897 has been awarded to Mr. J. LARMOR of St. John's College for his investigation on the "Physics of the æther" and other contributions to mathematical science.

THE works of the late Professor E. BELTRAMI are to be published by the faculty of science of the University of Rome. The edition, which is to be issued by subscription, will consist of three or four large volumes. A complete list of Professor BELTRAMI'S published papers appeared in the *Annali di Matematica* for March, 1900.

THE recent catalogue, No. 127, of M. Hoepli, Milan, Italy, contains nearly three thousand titles of memoirs and treatises in mathematics and physics.

PROFESSOR R. LIPSCHITZ, of Bonn, has been elected a correspondent of the section of geometry and Professor P. DUHEM correspondent of the section of mechanics of the Paris academy of sciences.

PROFESSOR F. KLEIN, of Göttingen, has been elected corresponding member of the Vienna academy of sciences. At the recent celebration of its five-hundredth anniversary, the University of Cracow conferred on Professor Klein an honorary doctorate.

DR. J. H. GORE, professor of mathematics and geodesy at the Columbian University, has been appointed juror-in-chief of the Court of Appeal of the Paris Exposition.

PROFESSOR A. R. FORSYTH has been elected an honorary member of the Edinburgh Royal Society.

DR. ARNOLD EMCH, recently professor of graphic mathematics in the Kansas State Agricultural College, has been appointed to an assistant professorship of mathematics in the University of Colorado.

DR. LOUIS T. MORE, of the University of Nebraska, has been elected professor of physics in the University of Cincinnati.

DR. JACOB WESTLUND, of Yale University, has been appointed instructor in mathematics in Purdue University.

DR. H. FEHR, docent at the University of Geneva, has been promoted to a full professorship in geometry and algebra at the same university.

DR. KARL BOEHM has been made docent in mathematics, at the University of Heidelberg.

MR. W. B. FITE and Mr. H. B. KUHN, recent holders of fellowships, have been appointed assistants in the department of mathematics at Cornell University.

PROFESSOR C. S. VENABLE, emeritus professor of mathematics in the University of Virginia, died August 11, 1900, at his home in Charlottesville, Va., at the age of 73 years. His education was received at Hampden Sidney College and the University of Virginia, and at the Universities of Berlin and Bonn. He served in the Confederate army with distinction throughout the civil war, most of the time as a member of Gen. Lee's staff. Before the war he was successively a member of the faculties of Hampden Sidney College, the University of Georgia, and the College of South Carolina; and since the war he served continuously as professor of mathematics in the University of Virginia until in 1896 failing health induced him to resign.

As the author of a series of well known text books on mathematics and by his work at the University of Virginia, he exerted a profound influence upon mathematical instruction throughout the Southern States. As a pupil of Encke and Argelander, he became deeply interested in astronomy. In 1860 he observed the total eclipse of the sun in Labrador, and more recently it was mainly through his exertions that the establishment and endowment of the Leander McCormick Observatory was secured.

PROFESSOR J. E. KEELER, director of the Lick Observatory, died at San Francisco, August 12, 1900, aged 43 years.

THE death is announced of Dr. G. BRUNEL, professor of mechanics and dean of the faculty of sciences of Bordeaux.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ANDOYER (H.). Leçons sur la théorie des formes et la géométrie analytique supérieure, à l'usage des étudiants des facultés des sciences. (En 2 volumes.) Vol. I. Paris, Gauthier-Villars, 1900. 8vo. 6 + 508 pp. Fr. 15.00

ARCAIS (F. D'). Corso del calcolo infinitesimale. Vol. II, parte 1. 2a edizione, con aggiunte e modificazioni. Padova, Draghi, 1900. 8vo. 368 pp. Prezzo dell'opera completa in due volumi Fr. 18.00

BACHMANN (P.). See ENCYKLOPÄDIE.

BALL (R. S.). A treatise on the theory of screws. London and New York, Macmillan, 1900. 8vo. 19 + 544 pp. (Cambridge University Press.) Cloth. $5.50

BÔCHER (M.). See ENCYKLOPÄDIE.

BOUTROUX (P.). L'imagination et les mathématiques, selon Descartes. Paris, Alcan, 1900. 8vo. 51 pp. (Bibliothèque de la Faculté des lettres de l'Université de Paris, No. X.) Fr. 2.00

BURKHARDT (H.). See ENCYKLOPÄDIE.

BURONI (S.). See LAVAGGI.

CAHEN (E.). Eléments de la théorie des nombres : Congruences ; formes quadratiques ; nombres incommensurables ; questions diverses. Paris, Gauthier-Villars, 1900. 8vo. 8 + 403 pp. Fr. 12.00

CALINON (A.). Etude de géométrie numérique. Paris, Gauthier-Villars, 1900. 8vo. 35 pp. Fr. 2.00

COMBEROUSSE (C. DE). Cours de mathématiques, à l'usage des candidats à l'Ecole polytechnique, à l'Ecole normale supérieure, à l'Ecole centrale des arts et manufactures. Vol. I : Aritnmétique et algèbre élémentaire. 4e édition, revue et augmentée. Paris, Gauthier-Villars, 1900. 8vo. Fr. 10.00

COMPLETE INDEX of all the papers printed in the *Proceedings of the London Mathematical Society*, vols. 1-30. London, Hodgson, 1900. 12mo. 32 pp.

COX (H.). Rudimentary treatise on integral calculus. London, 1900. 8vo. 136 pp. Cloth. 1s. 6d.

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. (In 7 Bänden.) Vol. I, Heft 5, enthaltend : Gleichungs-ysteme, von E. Netto und K. Th. Vahlen ; Endliche Gruppen linearer Substitutionen, von A. Wiman ; Niedere Zahlentheorie, von P. Bachmann ; Theorie der algebraischen Zahlkörper, von D. Hilbert ; Arithmetische Theorie algebraischer Grössen, von G. Landsberg ; Komplexe Multiplikation, von H. Weber. Leipzig, Teubner, 1900. 8vo. Pp. 513-720. M. 6.40

—— Vol. II, Heft 4, enthaltend: Kontinuierliche Transformationsgruppen, von L. Maurer und H. Burkhardt; Randwertaufgaben bei gewöhnlichen Differentialgleichungen, von M. Bôcher; Potentialtheorie (Theorie der Laplace-Poisson'schen Differentialgleichung), von H. Burkhardt und W. F. Meyer; Randwertaufgaben in der Theorie der partiellen Differentialgleichungen, von A. Sommerfeld. Leipzig, Teubner, 1900. 8vo. Pp. 401–560. M. 4.80

FAERBER (C.). Irrationale Zahlen und Verhältnisse inkommensurabler Grössen. (Progr.) Berlin, Gaertner, 1900. 4to. 33 pp. M. 1.00

FAJON (H.). Complément d'algèbre élémentaire. Variations des fonctions du premier degré, du second degré et bicarrées, à l'usage des candidats au baccalauréat, aux Ecoles de Saint-Cyr, navale et à l'Institut agronomique. Paris, Belin [1900]. 8vo. 69 pp.

FREYCINET (C. DE). Essais sur la philosophie des sciences: Analyse, mécanique. 2e édition. Paris, Gauthier-Villars, 1900. 8vo. 13 + 336 pp. Fr. 6.00

FRICKE (R.). Kurzgefasste Vorlesungen über verschiedene Gebiete der höheren Mathematik mit Berücksichtigung der Anwendungen. Analytisch-funktionentheoretischer Teil. Leipzig, Teubner, 1900. 8vo. 9 + 520 pp. Cloth. M. 14.00

GUIMARAES (R.). Les mathématiques en Portugal au XIX. siècle. Aperçu historique et bibliographique. Coimbre, 1900. 4to. 167 pp. Fr. 15.00

HAGEN (J. G.). Synopsis der höheren Mathematik. Vol. III: Differential- und Integralrechnung. (In 6 Lieferungen.) Lieferung 2. Berlin, Dames, 1900. 4to. Pp. 65–128. M. 5.00

HERTZER (H.). Die geometrischen Grundprinzipien der Parallel-Projektion. 3te Auflage. Berlin, Spaeth, 1900. 8vo. 4 + 68 pp., 3 plates. Cloth. M. 2.50

HILBERT (D.). See ENCYCLOPÄDIE.

JAHNKE (E.). Ueber dreifach perspektivische Dreiecke in der Dreiecksgeometrie. (Progr.) Berlin, Gaertner, 1900. 4to. 26 pp. M. 1.00

JAHRBUCH über die Fortschritte der Mathematik. Herausgegeben von E. Lampe. Vol. 29: Jahrgang 1898. (In 3 Heften.) Heft 1. Berlin, Reimer, 1900. 8vo. 416 pp. M. 13.00

KILLING (W.). Lehrbuch der analytischen Geometrie in homogenen Koordinaten. (In 2 Teilen.) Teil I: Die ebene Geometrie. Paderborn, Schöningh, 1900. 8vo. 13 + 220 pp. M. 4.00

KOCH (H. VON). Föreläsningar öfver teorin om transformationsgruppen. Stockholm, 1900. 4to. 189 pp. M. 5.60

LANDSBERG (G.). See ENCYKLOPÄDIE.

LANGE (J.). Synthetische Geometrie der Kegelschnitte nebst Uebungsaufgaben für die Prima höherer Lehranstalten. 2te Auflage. Berlin, Müller. 8vo. 68 pp. Cloth. M. 1.50

LAVAGGI. Calcolo infinitesimale; lezioni [dettate nell'anno] 1899–1900 nella r. università di Parma, compilate per cura di S. Buroni. Parma, Zafferri, 1900. 8vo. Pp. 1–432.

LAMPE (E.). See JAHRBUCH.

LORENTZ (H. A.). Lehrbuch der Differential- und Integralrechnung und der Anfangsgründe der analytischen Geometrie. Mit besonderer Berücksichtigung der Bedürfnisse der Studierenden der Naturwissenschaften. Unter Mitwirkung des Verfassers übersetzt von G. C. Schmidt. Leipzig, Barth, 1900. 8vo. 7 + 476 pp. M. 10.00

LUDWIG (B. VON). Ueber die Nothwendigkeit der Beschränkung des Jacobi'schen Umkehrproblems auf Abel'sche Integrale erster Gattung. (Diss.) Halle, 1900. 8vo. 94 pp.

LÜBSEN (H. B.). Einleitung in die Infinitesimalrechnung (Differential- und Integralrechnung). 8te Auflage. Leipzig, 1900. 8vo. 4 + 383 pp. M. 8.00

—— Ausführliches Lehrbuch der analytischen oder höheren Geometrie. 14te Auflage. Leipzig, 1900. 8vo. 4 + 214 pp. M. 4.00

MCGINNIS (M. A.). Universal solution for numerical and literal equations, by which roots of equations of all degrees can be expressed in terms of their coefficients. London, Sonnenschein, 1900. 12mo. 206 pp. 5s.

MAURER (L.). See ENCYKLOPÄDIE.

MEYER (W. F.). See ENCYKLOPÄDIE.

MICHEL (F.). Recueil de problèmes de géométrie analytique, à l'usage des élèves de mathématiques spéciales. Solutions des problèmes donnés aux concours d'admission à l'Ecole polytechnique de 1860 à 1900. Paris, Gauthier-Villars, 1900. 8vo. Fr. 6.00

MOEBIUS (P. J.). Ueber die Anlage zur Mathematik. Leipzig, Barth, 1900. 8vo. 7 + 331 pp, 51 portraits. M. 7.00

NETTO (E.). See ENCYKLOPÄDIE.

OPITZ (H. R. G.). Die Kramp-Laplace'sche Transcendente und ihre Umkehrung. (Progr.) Berlin, Gaertner, 1900. 4to. 29 pp. M. 1.00

OSTER (B.). Ueber partielle Differentialgleichungen zweiter Ordnung mit n unabhängigen Variabeln. Berlin, 1900. 8vo. 27 pp.

PICARD (E.). Sur le développement, depuis un siècle, de quelques théories fondamentales dans l'analyse mathématique. Conférences faites à Clark-University (Etats-Unis). Paris, Colin, 1900. 8vo. 91 pp. (Extrait de la *Revue générale des sciences.*)

PRANG (C.). Einführung in die Theorie und den Gebrauch der Determinanten. Berlin, Mayer und Müller, 1900. 8vo. 4 + 53 pp. Boards. M. 1.40

RICCI (G.). Lezioni di algebra complementare. Verona, Drucker, 1900. 8vo. 17 + 466 pp. Fr. 10.00

RIEMANN (B.). See WEBER (H.).

SCHEFFERS (G.). Einzelnes aus der Theorie der Curven und Flächen. (*Berichte der mathematisch-physischen Classe der K. Sächsischen Gesellschaft der Wissenschaften*, 8. Januar, 1900.) 12mo. 8 pp.

—— Funktionen der Abstände von festen Punkten. (*Mathematisch-naturwissenschaftliche Mitteilungen in Württemberg.*) Stuttgart, Metzler, 1900. 12mo. 17 pp.

SCHMIDT (G. C.). See LORENTZ (H. A.).

SOMMER (J.). Focaleigenschaften quadratischer Mannigfaltigkeiten im vierdimensionalen Raum. (*Mathematische Annalen*, vol. 53, pp. 113–160.) 8vo.

SOMMERFELD (A.). See ENCYKLOPÄDIE.

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